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DETERMINATION OF THE EFFECTIVE ELASTIC PROPERTIES FOR BIAXIALLY FIBER REINFORCED MATERIALS

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August 1967

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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This report was prepared by The Franklin Institute Research Laboratories under the terms of Contract DA 44-177-AMC-351(T).

The data contained herein are the result of research conducted to derive the upper and lower bounds of the effective elastic properties of biaxially fiber reinforced materials. The method used is based on bounding techniques on the geometrical model of fiber reinforced material, and the bounds obtained are generally very close.

The report has been reviewed by the U. S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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DETERMINATION OF THE EFFECTIVE ELASTIC PROPERTIES FOR BIAXIALLY FIBER REINFORCED MATERIALS

Final Report
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SUMMARY

Upper and lower bounds on the effective elastic properties of biaxially fiber reinforced material are derived. Two geometric models for the composite have been considered, and results are presented for several practical materials. In addition, bounds on the components of the shell stiffness tensor are obtained in terms of the bounds of the elastic moduli.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONT.)

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (CONT)

LIST OF ILLUSTRATIONS (CONT)

LIST OF ILLUSTRATIONS (CONT)

LIST OF ILLUSTRATIONS (CONT)

LIST OF TABLES

LIST OF SYMBOLS

$a_{\alpha\beta}$	Components of surface matrix.
$a^{\alpha\beta}$	Components of conjugate surface matrix, see Eq. (209).
$\bar{A}_i^{\alpha\beta\gamma\delta}$	Shifted components of elasticity tensor for the i^{th} layer.
$b_{\alpha\beta}$, b_β^α	Coefficients of the second fundamental form of the middle surface.
$B_n^{\nu\lambda\epsilon\kappa}$	See Eq. (200).
C_{ijkl}, C_{kl}^{ij}	Elastic moduli of biaxially reinforced material.
$C_i{}^{\alpha\beta}_{\gamma\delta}$	Elastic moduli of i^{th} layer.
C^*_{ijkl}	Effective elastic moduli of biaxially reinforced material.
C^*_{ij}	Non-zero elements of C^*_{ijkl}.
$\tilde{C}_{ijkl}$	See Eq. (79).
$\underline{C}^{*1}$	See Eq. (103).
$\underline{C}^{*2}$	See Eq. (98).
$\Delta\underline{C}$	See Eq. (88).
$\Delta\underline{C}^1$	See Eq. (90).
$\Delta\underline{C}^2$	See Eq. (91).
CCM	Composite Cylinder Model.
$D_n^{\alpha\beta\nu\lambda}$	Components of shell stiffness tensor.
e_{ij}, e^{ij}	Deviatoric components of strain tensor, ε_{ij}.
E	Young's modulus.
$E_o^{\nu\lambda\epsilon\kappa}$	See Eq. (199).
G	Shear modulus.
h	Shell thickness.
k	$(\lambda + G)$

LIST OF SYMBOLS (CONT)

K	Bulk modulus.
LM	Layer Model.
LSM	Layer Shell Model.
n	Outward normal at surface.
$N^{\alpha\beta}$	Symmetric stress resultants.
$M^{\alpha\beta}$	Symmetric stress couples.
$p_{\alpha\beta}$	Shell strain measure, see Eq. (190).
$q_{\alpha\beta}$	Shell strain measure, see Eq. (191).
R	Least radius of curvature of shell.
(S)	Indicates quantity defined on a surface.
S_{ijkl}	Elastic compliances of biaxially reinforced material.
S^*_{ijkl}	Effective elastic compliances of biaxially reinforced material.
S^*_{ij}	Non-zero elements of S^*_{ijkl}.
$\tilde{S}_{ijkl}$	See Eq. (80).
$\underline{S}^{*1}$	See Eq. (104).
$\underline{S}^{*2}$	See Eq. (99).
$\Delta\underline{S}$	See Eq. (89).
$\Delta\underline{S}^1$	See Eq. (92).
$\Delta\underline{S}^2$	See Eq. (93).
T	Boundary traction.
u_i	Surface displacement.
V	Total volume.
V_c	Volume occupied by composite cylinders.
V_R	Difference between total volume and volume of composite cylinders.

LIST OF SYMBOLS (CONT)

v_f	Ratio of fiber volume to total volume.
v_m	Ratio of matrix volume to total volume.
W^ε	Energy density in terms of strains.
W^σ	Energy density in terms of stresses.
x_1, x_2, x_3	Cartesian coordinate system defined by Figure 1.
x_1', x_2', x_3'	Cartesian coordinate system defined by Figure 2.
x_1'', x_2'', x_3''	Cartesian coordinate system defined by Figure 2.

GREEK SYMBOLS

α_{ij}	Direction cosines.
α'_{ij}	Same as α_{ij}.
α''_{ij}	Direction cosines with ϕ replaced by $(180-\phi)$.
β_α	Components of rotation.
δ^α_β	Kronecker delta.
ε_{ij}	Strain tensor.
η	G_f/G_m
λ	$\frac{\nu E}{(1+\nu)(1-2\nu)}$
μ	$\lvert \mu^\alpha_\beta \rvert$
μ^α_β	Shifters defined by Eq. (201).
$\bar{\mu}^\beta_\gamma$	Inverse shifters.
ν_{ij}	Poisson's ratio.
ξ	Distance from a reference surface.

GREEK SYMBOLS (CONT)

ξ_i	Distance from a reference surface to the inner surface of the i^{th} layer.
σ_{ij}	Stress tensor.
ϕ	Reinforcement angle, i.e., the angle between the fiber direction and the x_1 axis.
ψ	See Eq. (49b).

SUBSCRIPTS

a	Axial.
c	Composite cylinder.
f	Fiber.
i	Vector component.
ijkl	Tensor components. The range of subscripts is 1, 2, 3 and a repeated subscript indicates summation.
m	Matrix.
(n)c	Nth composite cylinder.
R	Remaining volume between volume of composite cylinders and total volume.
t	Transverse.
ε	Strain.
σ	Stress.

SUPERSCRIPTS

*	Effective value.
(+)	Upper bound.

SUPERSCRIPTS (CONT.)

(-)	Lower bound.
A'	Value of A in x_1', x_2', x_3' system of coordinates.
A''	Value of A in x_1'', x_2'', x_3'' system of coordinates.
A°	Spatially constant value of A.
c	Composite cylinder.
ε	Strain.
σ	Stress.

OTHER SYMBOLS

$'A$	Form of A with reinforcement angle equal to ϕ
$''A$	Form of A with reinforcement angle equal to $(180-\phi)$.
$\bar{A}$	Average value of A in a volume.
$\tilde{A}$	Admissable field form of A.
$A_{i,j}$	$\dfrac{\partial A_i}{\partial x_j}$
$A\|_{\alpha}$	Indicates covariant differentiation with respect to the surface metric.
$\underline{x}$	Abbreviation for x_1, x_2, x_3.
< >	Denotes physical components.

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1. INTRODUCTION

Theoretical study of the elastic behavior of fiber reinforced materials has to date been devoted primarily to uniaxially reinforced materials. Hashin and Rosen [1]* introduced the composite cylinder assemblage model as an idealization of a uniaxially reinforced material. By use of variational methods, four of the five independent elastic moduli were exactly calculated on the basis of this model, and lower and upper bounds were given for a fifth modulus. In many cases, experimental results are quite close to the theoretical predictions [2,3].

The problem has also been treated from a very general point of view by Hill [4] and Hashin [5]. In these investigations, no special internal geometry is assumed; the fiber cross sections may have arbitrary shapes and their positions are random. It should be realized that under these circumstances much less is known about the internal geometry of the composite than in the case of the composite cylinder assemblage used in [1], for in the latter case the fiber cross sections must be circular and their arrangement is of a very special nature. Consequently, the investigations [4,5] did not permit exact calculation of any effective modulus, since the general model used did not contain enough information for unique determination of effective moduli. Therefore, the approach consisted of bounding of the effective moduli.

The investigations [4,5] should be considered as a first step towards the solution of a general and difficult problem; namely, estimation of effective moduli for internal geometry which is specified in a statistical manner. Since the bounds are based only on very rudimentary geometrical information, i.e, phase volume fractions, it is not surprising that for the elevated fiber to matrix stiffness ratios usually encountered in practice, not all the bounds give sufficiently close estimates to be of practical importance. Therefore, in what follows, the results obtained in [1] will be used for effective moduli of uniaxially reinforced materials.

A number of other investigations of the problem are to be found in the literature. However, these are based on very crude approximations and thus do not seem to be sufficiently reliable. For a general comprehensive survey of work on properties of uniaxially reinforced materials, see [6].

Practical experience and theoretical study of uniaxially reinforced materials reveal that these materials are very stiff and strong in fiber directions. However, in directions transverse to the fibers,

*Numbers in brackets refer to appended references.

both stiffness and strength are an order of magnitude smaller. In many applications, such as plate and shell constructions, stiffness and strength of comparable magnitudes are required in all directions. This may be achieved by constructing a material with two or more directions of reinforcement. The present investigation is concerned with the theoretical prediction of the elastic properties of a biaxially fiber reinforced material. The method used is based on bounding techniques on the geometrical model of a fiber reinforced material which has been introduced in [1]. It will be seen that the bounds obtained are mostly very close together and thus give very good estimates of the effective elastic properties.

The results are then exploited for analysis of elastic behavior of biaxially reinforced shells.

2. DEFINITION OF EFFECTIVE ELASTIC MODULI

The material under consideration is a biaxially fiber reinforced material. It is composed of layers which are assumed to be of equal thickness h. In each layer the fibers are in one direction. The reinforcement direction alternates from layer to layer. The material is referred to a coordinate system x_1, x_2, x_3, in which x_2 is normal to the layers and x_1, x_3 bisect the angles formed by alternating reinforcing fibers (Figure 1). The angles between fiber directions and the x_1 axis are ϕ and $(180 - \phi)$, respectively. It is assumed that in each layer the number of fibers through the thickness is very large and also that in the material specimen the number of layers is very large. The matrix and fiber materials are assumed to be linear elastic and isotropic. The problem then is to predict the elastic properties of the biaxially reinforced material in terms of elastic moduli of the constituents, their volume fractions or additional geometrical characteristics, and the angle of reinforcement ϕ.

The effective elastic moduli of a statistically homogeneous elastic composite material are defined in terms of stress and strain averages. Let a large specimen be subjected to surface displacements of the form

$$u_i^o(S) = \varepsilon_{ij}^o x_j, \tag{1}$$

where ε_{ij}^o are constant. Then the volume average strains $\bar{\varepsilon}_{ij}$, defined by

$$\bar{\varepsilon}_{ij} = \frac{1}{V} \int \varepsilon_{ij} \ (\underline{x}) dV, \tag{2}$$

are given by

$$\bar{\varepsilon}_{ij} = \varepsilon_{ij}^o. \tag{3}$$

The average stresses are necessarily linearly related to the average strains. Write

$$\bar{\sigma}_{ij} = C_{ijkl}^* \bar{\varepsilon}_{kl} = C_{ijkl}^* \varepsilon_{kl}^o. \tag{4}$$

The coefficients C_{ijkl}^* are defined as the effective elastic moduli. Dually, let the specimen be subjected to boundary tractions of the form

$$T_i^o(S) = \sigma_{ij}^o n_j, \tag{5}$$

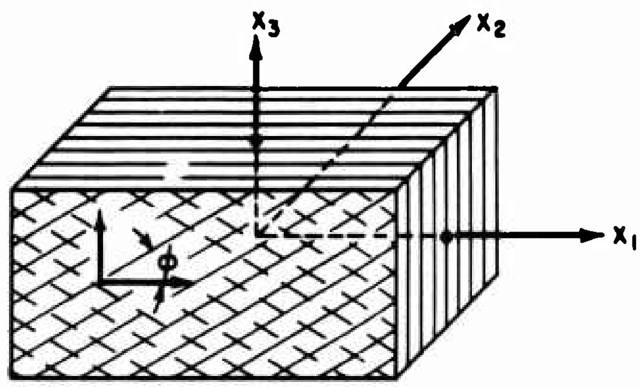

Figure 1. Geometrical Model of the Material.

where n_j are the components of the outward normal and σ^o_{ij} are constant stresses. Then

$$\bar{\sigma}_{ij} = \sigma^o_{ij}. \tag{6}$$

The necessarily linear relation between the average stresses and strains is now written in the form

$$\bar{\varepsilon}_{ij} = S^*_{ijkl}\, \sigma^o_{kl}. \tag{7}$$

The coefficients S^*_{ijkl} are defined as the effective elastic compliances. For a very large specimen, the tensors C^*_{ijkl} and S^*_{ijkl} are reciprocal, and this will be assumed forthwith.

The effective elastic moduli and compliances can also be defined in terms of stored elastic energy. When (1) is prescribed, it may be shown [1] that the strain energy W^ε stored in a large specimen is

$$W^\varepsilon = \frac{1}{2} C^*_{ijkl}\, \varepsilon^o_{ij} \varepsilon^o_{kl}. \tag{8}$$

Here the specimen volume has been taken as unit volume without loss of generality. Dually, when (5) is prescribed, the stress energy (here and in what follows, elastic energy in terms of strains is termed strain energy; elastic energy in terms of stresses, stress energy) is given by [1]

$$W^\sigma = \frac{1}{2} S^*_{ijkl}\, \sigma^o_{ij}\, \sigma^o_{kl}. \tag{9}$$

Expressions (8) and (9) may be regarded as the strain and stress energy densities of the composites.

In general, the number of effective elastic moduli is at most 21. Material symmetry will reduce this number. It is clear from Figure 1 that the present material has three planes of elastic symmetry, i.e., the $x_1\, x_2$, $x_2\, x_3$, and $x_3\, x_1$ planes. Consequently, the material is orthotropic and has nine independent effective elastic moduli or compliances. The stress-strain relation (9) then assumes the following form [7]:

$$\bar{\sigma}_{11} = C^*_{11}\, \bar{\varepsilon}_{11} + C^*_{12}\, \bar{\varepsilon}_{22} + C^*_{13}\, \bar{\varepsilon}_{23}, \qquad \text{(a)}$$

$$\bar{\sigma}_{22} = C^*_{12}\,\bar{\varepsilon}_{11} + C^*_{22}\,\bar{\varepsilon}_{22} + C^*_{23}\,\bar{\varepsilon}_{33}, \qquad \text{(b)}$$

$$\bar{\sigma}_{33} = C^*_{13}\,\bar{\varepsilon}_{11} + C^*_{23}\,\bar{\varepsilon}_{22} + C^*_{33}\,\bar{\varepsilon}_{33}, \qquad \text{(c)} \qquad (10)$$

$$\bar{\sigma}_{12} = 2C^*_{44}\,\bar{\varepsilon}_{12}, \qquad \text{(a)}$$

$$\bar{\sigma}_{23} = 2C^*_{55}\,\bar{\varepsilon}_{23}, \qquad \text{(b)}$$

$$\bar{\sigma}_{31} = 2C^*_{66}\,\bar{\varepsilon}_{31}, \qquad \text{(c)} \qquad (11)$$

where six-by-six matrix notation has been used for the C^*_{ijkl}. It is evident that the coefficients in (11 a,b,c) are shear moduli. Therefore, these will be given the following notation:

$$C^*_{44} = G^*_{12}, \qquad \text{(a)}$$

$$C^*_{55} = G^*_{23}, \qquad \text{(b)}$$

$$C^*_{66} = G^*_{31}. \qquad \text{(c)} \qquad (12)$$

The stress-strain relation (7) can be similarly written:

$$\bar{\varepsilon}_{11} = S^*_{11}\,\bar{\sigma}_{11} + S^*_{12}\,\bar{\sigma}_{22} + S^*_{13}\,\bar{\sigma}_{33}, \qquad \text{(a)}$$

$$\bar{\varepsilon}_{22} = S^*_{12}\,\bar{\sigma}_{11} + S^*_{22}\,\bar{\sigma}_{22} + S^*_{23}\,\bar{\sigma}_{33}, \qquad \text{(b)}$$

$$\bar{\varepsilon}_{33} = S^*_{13}\,\bar{\sigma}_{11} + S^*_{23}\,\bar{\sigma}_{22} + S^*_{33}\,\bar{\sigma}_{33}, \qquad \text{(c)} \qquad (13)$$

$$\bar{\varepsilon}_{12} = 2S^*_{44}\,\bar{\sigma}_{12}, \qquad \text{(a)}$$

$$\bar{\varepsilon}_{23} = 2S^*_{55}\,\bar{\sigma}_{23}, \qquad \text{(b)}$$

$$\bar{\varepsilon}_{31} = 2S^*_{66}\,\bar{\sigma}_{31}. \qquad \text{(c)} \qquad (14)$$

The three-by-three symmetric matrices C^*_{ij} and S^*_{ij} $(i, j = 1, 2, 3)$ are reciprocal. It is common practice to write the S^*_{ij} in (13) in terms of Young's moduli and Poisson's ratios [7]. Thus,

$$
\begin{aligned}
S^*_{11} &= \frac{1}{E^*_1} && \text{(a)} \\
S^*_{22} &= \frac{1}{E^*_2}\,, && \text{(b)} \\
S^*_{33} &= \frac{1}{E^*_3}\,, && \text{(c)}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
S^*_{12} &= -\frac{\nu^*_{12}}{E^*_1} = -\frac{\nu^*_{21}}{E^*_2}\,, && \text{(a)} \\
S^*_{23} &= -\frac{\nu^*_{23}}{E^*_2} = -\frac{\nu^*_{32}}{E^*_3}\,, && \text{(b)} \\
S^*_{31} &= -\frac{\nu^*_{31}}{E^*_3} = -\frac{\nu^*_{13}}{E^*_1}\,. && \text{(c)}
\end{aligned}
\tag{16}
$$

To explain the meaning of these, assume, for example, uniaxial average stress $\bar{\sigma}_{11}$ and all others vanish. Then

$$
\bar{\varepsilon}_{11} = \frac{\bar{\sigma}_{11}}{E^*_1}\,, \quad \bar{\varepsilon}_{22} = -\frac{\nu^*_{12}}{E^*_1}\,\bar{\sigma}_{11}, \quad \bar{\varepsilon}_{33} = -\frac{\nu^*_{13}}{E^*_1}\,\bar{\sigma}_{11},
$$

and the general stress-strain law (13) can be obtained by superposition of the normal stresses in the coordinate directions.

3. GEOMETRICAL MODELS OF BIAXIALLY FIBER REINFORCED MATERIAL

For the purpose of prediction of effective elastic properties, two geometrical models of the composite will be considered. The first of these is termed the layer model (LM); and the second, the composite cylinder model (CCM).

LAYER MODEL

The material is composed of homogeneous elastic layers of equal thickness. Each layer has the elastic properties of a uniaxially fiber reinforced material. Such materials are transversely isotropic, the fiber direction being the axis of elastic symmetry. As a result of this symmetry, such a material has five independent, effective, elastic moduli.

It should be borne in mind that in order to bring out this symmetry, it is necessary to use a coordinate system, one of whose axes is in the fiber direction while the other two are normal to it. These coordinate systems are shown in Figure 2, with respect to the two different fiber layers.

In terms of the model described above, the problem takes on the following form: given the effective elastic moduli of a layer, or bounds thereon, with respect to the primed and double-primed coordinate systems, find the effective elastic moduli, or bounds thereon, for the layer composite.

COMPOSITE CYLINDER MODEL

In this model, more detailed account is taken of the internal geometry of the composite than in the LM. It is assumed that each layer is described by the composite cylinder assemblage which has been introduced in [1]. To describe such an assemblage, assume that one has at his disposal composite cylinders, each of which consists of a circular cylindrical fiber and a concentric circular matrix shell, perfectly bonded together. In each cylinder, the ratio between fiber radius and outer matrix shell ratio is the same (Figure 3(a)). The volume of each layer is imagined to be filled out progressively with parallel composite cylinders of decreasing outer radii, (Figure 3(b)). In the limit, the whole volume is thus filled out, and the composite analysis then becomes based on the analysis of a single composite cylinder.

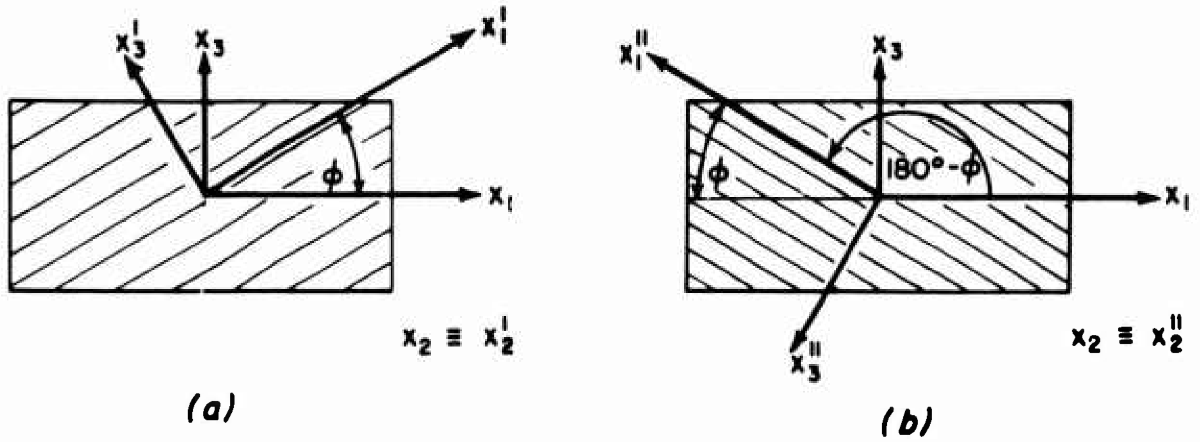

Figure 2. Reference Frames.

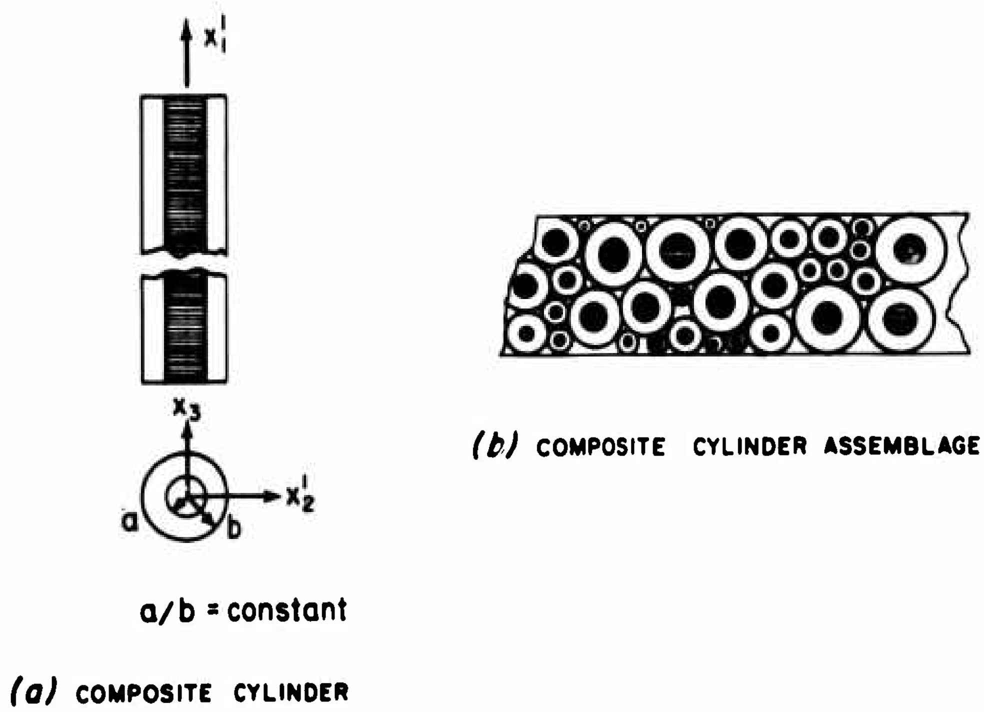

Figure 3. Composite Cylinder Model.

4. VARIATIONAL BOUNDING METHODS FOR EFFECTIVE ELASTIC MODULI

As has been shown in Section 2, the determination of effective elastic moduli is equivalent to the determination of stress averages for a boundary condition of type (1), or of strain average for a boundary condition of type (5). In general, in order to find an average, the detailed field must first be computed. It is clear that field determination in biaxially fiber reinforced materials is an intractable problem.

This difficulty can be avoided to a large extent by variational bounding techniques. Here, the aim is to find lower and upper bounds for the effective moduli and compliances, instead of attempting to compute them.

The possibility of variational bounding hinges upon the results (8) and (9), and the variational principles of the theory of elasticity. According to (8) and (9), the strain or stress energy is given in terms of strains or stresses known from the boundary conditions, and the effective elastic moduli or compliances. The elastic energy can be bounded by use of the variational principles of elasticity, and thus bounds are obtained on combinations of C^*_{ijkl} or S^*_{ijkl}.

BOUNDING METHOD FOR LAYER MODEL

Let a displacement field of type (1) be prescribed on the boundary. An <u>admissible displacement field</u>, $\tilde{u}_i(\underline{x})$, for the principle of minimum potential energy (see [8], for example) is defined as a continuous field which satisfies the boundary conditions. The simplest admissible field for the present case is

$$\tilde{u}_i(\underline{x}) = \varepsilon^o_{ij} x_j, \tag{17}$$

which obviously satisfies all the requirements.

Now compute the integral

$$\tilde{W}^\varepsilon = \frac{1}{2} \int_V C_{ijkl} (\underline{x}) \, \tilde{\varepsilon}_{ij} \, \tilde{\varepsilon}_{kl} \, dV \tag{18}$$

throughout the LM specimen. Here,

$$\tilde{\varepsilon}_{ij} = \frac{1}{2} (\tilde{u}_{i,j} + \tilde{u}_{j,i}) \tag{19}$$

and $C_{ijkl}(\underline{x})$ are the space variable elastic moduli. In the present case C_{ijkl} assume only two different values because there are only two kinds

of layers. Denote these by $'C_{ijkl}$ for the layers in which the fibers are inclined at angle ϕ with respect to x_1, and by $''C_{ijkl}$ for the layers in which the fibers are inclined at angle $(180 - \phi)$ with respect to x_1 (Figure 2). $'C_{ijkl}$ and $''C_{ijkl}$ are the elastic moduli of the transversely isotropic layers referred to in the specimen coordinate system x_1, x_2, x_3. Since the layer moduli are defined with respect to the coordinate systems x_1', x_2', x_3' or x_1'', x_2'', x_3'' (Figure 2), the values of $'C_{ijkl}$ and $''C_{ijkl}$ are defined by fourth-rank tensor transformation rules.

Now in the present case, from (17) and (19),

$$\tilde{\varepsilon}_{ij} = \varepsilon^{o}_{ij} \; . \tag{20}$$

Assuming equal volumes of different layers, and substituting (20) into (18), the integral (18) reduces to

$$2\tilde{W}^{\varepsilon} = \frac{1}{2} \; [{}'C_{ijkl} + {}''C_{ijkl}] \; \varepsilon^{o}_{ij} \; \varepsilon^{o}_{kl} \; . \tag{21}$$

From the principle of minimum potential energy,

$$W^{\varepsilon} \leq \tilde{W}^{\varepsilon} \; . \tag{22}$$

Inserting (21) and (8), respectively, into (22), one finds

$$C^{*}_{ijkl} \; \varepsilon^{o}_{ij} \; \varepsilon^{o}_{kl} \leq \frac{1}{2} \; ({}'C_{ijkl} + {}''C_{ijkl}) \; \varepsilon^{o}_{ij} \; \varepsilon^{o}_{kl} \; . \tag{23}$$

One can now proceed in like manner with the specimen loaded by tractions of type (5). Here an <u>admissible stress system</u> is needed in conjunction with the principle of minimum complementary energy [8]. An admissible stress system, $\tilde{\sigma}_{ij}(\underline{x})$, is defined as one which satisfies equilibrium, traction continuity, and the boundary conditions. The simplest $\tilde{\sigma}_{ij}$ in the present case is the constant stress system.

$$\tilde{\sigma}_{ij} = \sigma^{o}_{ij} \; . \tag{24}$$

Now compute the integral

$$\tilde{W}^{\sigma} = \frac{1}{2} \int_V S_{ijkl}(\underline{x}) \; \tilde{\sigma}_{ij} \; \tilde{\sigma}_{kl} \; dV \; , \tag{25}$$

where $S_{ijkl}(\underline{x})$ are the space variable compliances, which again assume only two values $'S_{ijkl}$ and $''S_{ijkl}$ in the different layers. They are the

reciprocals of $'C_{ijkl}$ and $''C_{ijkl}$, respectively. Proceeding as before for unit volume, (25) assumes the form

$$2\tilde{W}^{\sigma} = \frac{1}{2}\,('S_{ijkl} + ''S_{ijkl})\,\sigma^{o}_{ij}\,\sigma^{o}_{kl}\ . \tag{26}$$

Because of the principle of minimum complementary energy,

$$W^{\sigma} \leq \tilde{W}^{\sigma}. \tag{27}$$

Inserting (9) and (26) into (27), it is found that

$$S^{*}_{ijkl}\,\sigma^{o}_{ij}\,\sigma^{o}_{kl} \leq \frac{1}{2}\,('S_{ijkl} + ''S_{ijkl})\,\sigma^{o}_{ij}\,\sigma^{o}_{kl} \tag{28}$$

BOUNDING METHOD FOR COMPOSITE CYLINDER MODEL

Let the volume filled out by composite cylinders in the CCM be denoted by V_C and the remaining volume by V_R. Thus, by hypothesis, V_R may by made indefinitely small and in the limit

$$V_C \rightarrow V. \tag{29}$$

Let the biaxially reinforced specimen be subjected to a displacement of type (1) on the boundary. An admissible displacement field $\tilde{u}_i$ is now constructed as follows: The field (17) is imposed throughout V_R. The composite cylinders are then subjected to (17) on their boundaries. This defines an elasticity boundary value problem for a composite cylinder. Since by construction all composite cylinders are geometrically similar, the same solution is valid for each of them. Let the displacement in a composite cylinder thus obtained be denoted by $u^{c}{}_{i}$. Then the admissible displacement field for the composite is

$$\tilde{u}_i = u^{c}_{i} \quad \text{in } V_c, \qquad \text{(a)}$$

$$\tilde{u}_i = \varepsilon^{o}_{ij}\,x_j \quad \text{in } V_R. \qquad \text{(b)} \tag{30}$$

Obviously this field satisfies continuity and (1) on the boundary and is thus admissible.

Now (18) is computed using (30). The result can be written in the form

$$\tilde{W}^{\varepsilon} = \sum^{n} W^{\varepsilon}_{(n)c} + W^{\varepsilon}_{R}\ , \tag{31}$$

where $W^{\varepsilon}_{(n)c}$ is the strain energy in the nth composite cylinder subjected to (1) on its boundary and W^{ε}_{R} is (18) for V_R. Since the strains associated with (b) are constant (ε^{o}_{ij}), W^{ε}_{R} is proportional to V_R. Since V_R disappears in the limit, so does W^{ε}_{R}. Thus, in the limit

$$\tilde{W}^{\varepsilon} = \sum_{n=1}^{n=N\to\infty} W^{\varepsilon}_{(n)c} \,. \tag{32}$$

Because of the geometrical similarity, the strain energy $W^{\varepsilon}_{(n)c}$ in any composite cylinder can be written in the form

$$W^{\varepsilon}_{(n)c} = w^{\varepsilon} V_{(n)c} \tag{33}$$

where w^{ε} is one and the same number for all cylinders and is a function of ε^{o}_{ij}, the elastic moduli of fibers and matrix and the ratio between the radii a/b.

Therefore, (32) assumes the form

$$\tilde{W}^{\varepsilon} = w^{\varepsilon} \sum_{n=1}^{n=N\to\infty} V_{(n)c} = w^{\varepsilon}V. \tag{34}$$

Now from (8), (34), and the principle of minimum potential energy,

$$\frac{1}{2} C^{*}_{ijkl} \; \varepsilon^{o}_{ij} \; \varepsilon^{o}_{kl} \leq w^{\varepsilon} \tag{35}$$

where V has been taken as unit value. It should be remembered that in order to find w^{ε}, only one composite cylinder has to be analyzed.

A similar procedure is now used with the admissible stresses. For boundary conditions of type (5), the stress system σ^{o}_{ij} is imposed on V_R. Then the composite cylinders are subject to the tractions

$$T_i = \sigma^{o}_{ij} \; n_j \tag{36}$$

on their boundaries. The resulting stresses inside the composite cylinders are σ^{o}_{ij}. The admissible stress system in the composite specimen is then

$$\tilde{\sigma}_{ij} = \sigma^{o}_{ij} \text{ in } V_c, \qquad \text{(a)}$$

$$\tilde{\sigma}_{ij} = \sigma^{o}_{ij} \text{ in } V_R. \qquad \text{(b)} \tag{37}$$

Computation of (25) with (37) yields

$$\tilde{W}^{\sigma} = \sum^{n} W^{\sigma}_{(n)c} + W^{\sigma}_{R} , \tag{38}$$

where $W^{\sigma}_{(n)c}$ is the strain energy in the nth composite cylinder subjected to (36) on its boundary. Again in the limit as V_R becomes infinitesimal, so does W^{σ}_{R}. Then by reasoning completely analogous to that given above,

$$\tilde{W}^{\sigma} = w^{\sigma} V, \tag{39}$$

where $\tilde{W}^{\sigma}$ is defined by

$$\tilde{W}^{\sigma}_{(n)c} = w^{\sigma} V_{(n)c}. \tag{40}$$

Application of minimum complementary energy with (39) and (9), for unit composite volume, yields

$$\frac{1}{2} S^{*}_{ijkl} \quad \varepsilon^{o}_{ij} \quad \varepsilon^{o}_{kl} \geq w^{\sigma}. \tag{41}$$

It turns out that the variational bounding methods which have been described can be advantageously exploited only for bounding of the shear moduli G^{*}_{12} , G^{*}_{23} , G^{*}_{31} , defined in (11) and (12). The reason for this is that with the present orthotropic stress-strain laws (10), (11), (13), and (14), the energies on the left sides of (33), (28), (35), and (41) can be expressed in terms of one single effective modulus or compliance, only for the case of an average pure shear strain or stress. This is because only the shear stress-strain relations (11) and (14) are given in terms of one single modulus or compliance. In these circumstances, a bound on energy gives a bound on an effective modulus or compliance. However, in the case of average normal strains or stresses, the energies on the left sides of (23), (28), (35), and (41) involve several moduli or compliances. Thus, bounds on combinations of these are obtained, and it is a very difficult matter to deduce from these bounds on the individual moduli and compliances. These difficulties will be further explained below. Consequently, the variational bounding methods as described here will not be used for estimation of normal effective moduli and compliances.

5. RÉSUMÉ OF ELASTIC PROPERTIES OF UNIAXIALLY FIBER REINFORCED MATERIALS

Knowledge of the elastic properties of uniaxially fiber reinforced materials is of fundamental importance for the present study, since the biaxially reinforced composite is composed of uniaxially reinforced layers. Indeed, every effort will be made to base prediction of the elastic moduli of the biaxially reinforced material on the moduli of a uniaxially reinforced material.

Uniaxially fiber reinforced materials are transversely isotropic, the fiber direction being the axis of transverse isotropy. Let the fiber direction be x_1', while x_2', x_3' are the other two axes of the Cartesian reference system, Figure 2a. Similarly, an x_1'', x_2'', x_3'' system of axes may be taken for the other layer, Figure 2b.

The material has five independent, effective elastic moduli and the effective stress-strain law may be written in the following form [1]:

$$\bar{\sigma}_{11}' = C_{11}' \bar{\varepsilon}_{11}' + C_{12}' \bar{\varepsilon}_{22}' + C_{12}' \bar{\varepsilon}_{33}', \qquad \text{(a)}$$

$$\bar{\sigma}_{22}' = C_{12}' \bar{\varepsilon}_{11}' + C_{22}' \bar{\varepsilon}_{22}' + C_{23}' \bar{\varepsilon}_{33}', \qquad \text{(b)}$$

$$\bar{\sigma}_{33}' = C_{12}' \bar{\varepsilon}_{11}' + C_{23}' \bar{\varepsilon}_{22}' + C_{22}' \bar{\varepsilon}_{33}', \qquad \text{(c)} \qquad (42)$$

$$\bar{\sigma}_{12}' = 2C_{44}' \bar{\varepsilon}_{12}', \qquad \text{(a)}$$

$$\bar{\sigma}_{23}' = (C_{22}' - C_{23}') \bar{\varepsilon}_{23}', \qquad \text{(b)}$$

$$\bar{\sigma}_{31}' = 2C_{44}' \bar{\varepsilon}_{31}'. \qquad \text{(c)} \qquad (43)$$

Here, the overbar denotes average over the layer volume. The average strains may be expressed in terms of average stresses by relations similar to (42) and (43), via the effective elastic compliances.

Effective elastic moduli of technical importance will now be defined in terms of the moduli in (42) and (43):

Plane strain bulk modulus $$K_t = \frac{1}{2} (C_{22}' + C_{23}'), \qquad (44)$$

Transverse shear modulus $G_t = \frac{1}{2} (C'_{22} - C'_{23})$, (45)

Axial shear modulus $G_a = C'_{44}$, (46)

Axial Young's modulus $E_a = C'_{11} - \frac{C'^2_{12}}{K_t}$ (47)

Axial Poisson's ratio $\nu_a = \sqrt{\frac{C'_{11} - E_a}{K_t}} = \frac{C'_{12}}{2K_t}$, (48)

Transverse Young's modulus $E_t = \frac{4\, G_t\, K_t}{K_t + \psi\, G_t}$, (49 a)

where $\psi = 1 + \frac{4\, K_t\, \nu_a^2}{E_a}$, (49 b)

Transverse Poisson's ratio $\nu_t = \frac{K_t - \psi\, G_t}{K_t + \psi\, G_t}$. (50)

Also, note the relation

$$G_t = \frac{E_t}{2\,(1 + \nu_t)}, \quad (51)$$

which follows from (49) and (50) and is of the same form as an isotropic elasticity relation.

The physical significance of the moduli (49) and (50) will now be briefly summarized.

The modulus (44) corresponds to the plane strain situation

$$\bar{\varepsilon}'_{11} = 0, \quad \bar{\varepsilon}'_{22} = \bar{\varepsilon}'_{33},$$

$$\bar{\varepsilon}'_{12} = \bar{\varepsilon}'_{23} = \bar{\varepsilon}'_{31} = 0. \quad (52)$$

Then, from (42 b, c) and (44),

$$\bar{\sigma}'_{22} + \bar{\sigma}'_{33} = 2\, K_t\, (\bar{\varepsilon}'_{22} + \bar{\varepsilon}'_{33}). \tag{53}$$

The significance of (45) and (46) is obvious from (43 b) and (43 a, c), respectively. The shear modulus G_t is associated with shear in the transverse $x'_2\ x'_3$ plane, normal to the fibers. On the other hand, the shear modulus G_a is associated with shear either in the $x'_1\ x'_2$ plane or the $x'_1\ x'_3$ plane.

The Young's modulus (47) corresponds to the situation

$$\begin{aligned} &\bar{\sigma}'_{22} = \bar{\sigma}'_{33} = 0, \quad \bar{\sigma}'_{11} \neq 0, \\ &\bar{\sigma}'_{12} = \bar{\sigma}'_{23} = \bar{\sigma}'_{31} = 0, \end{aligned} \tag{54}$$

which represents uniaxial stress in fiber direction.

Then,

$$\bar{\varepsilon}'_{11} = \frac{\bar{\sigma}'_{11}}{E_a}\,. \tag{55}$$

Also, in the case (54),

$$\bar{\varepsilon}'_{22} = \bar{\varepsilon}'_{33} = -\,\nu_a\, \bar{\varepsilon}'_{11}, \tag{56}$$

which defines the Poisson's ratio (48).

The Young's modulus (49 a) is associated with either one of the loadings

$$\left.\begin{array}{ll} \bar{\sigma}'_{11} = \bar{\sigma}'_{22} = 0, & \bar{\sigma}'_{33} \neq 0 \\ \text{or} & \\ \bar{\sigma}'_{11} = \bar{\sigma}'_{33} = 0, & \bar{\sigma}'_{22} \neq 0 \end{array}\right\} \quad \bar{\sigma}'_{12} = \bar{\sigma}'_{21} = \bar{\sigma}'_{23} = \bar{\sigma}'_{31} = 0. \quad \begin{array}{l}\text{(a)}\\ \\ \text{(b)}\end{array} \tag{57}$$

Then for (57 a),

$$\bar{\varepsilon}'_{33} = \frac{\bar{\sigma}'_{33}}{E_t}\,, \qquad \bar{\varepsilon}'_{22} = -\nu_t\, \bar{\varepsilon}'_{33}\,, \qquad \text{(a)}$$

and for (57 b),

$$\bar{\varepsilon}'_{22} = \frac{\bar{\sigma}'_{22}}{E_t} , \qquad \bar{\varepsilon}'_{33} = -\nu_t \, \bar{\varepsilon}'_{22}, \qquad \text{(b)} \tag{58}$$

which also define the Poisson's ratio (50).

In addition, the modulus C'_{11} in (42 a) is of some physical interest. It is associated with the situation

$$\begin{aligned} &\bar{\sigma}'_{11} \neq 0 , \qquad \bar{\varepsilon}'_{22} = \bar{\varepsilon}'_{33} = 0, \\ &\bar{\sigma}'_{12} = \bar{\sigma}'_{23} = \bar{\sigma}'_{31} = 0 , \end{aligned} \tag{59}$$

which is the case of a specimen uniaxially loaded while transverse deformation is prevented by a rigid bonded enclosure.

Known expressions for effective elastic moduli of a uniaxially fiber reinforced material, analyzed on the basis of the composite cylinder model (CCM) will now be given. Quantities for matrix material will be given the subscript m, and quantities for fibers will be given the subscript f. The matrix fibers are assumed to be elastic isotropic. The volume fractions are v_b, v_f, $v_b + v_f = 1$. Isotropic elastic moduli used are G - shear modulus, ν - Poisson's ratio, E - Young's modulus, and $k = \lambda + G$ - plane strain bulk modulus.

$$K_t = k_m \left[1 + v_f \Big/ \left(\frac{k_m}{k_f - k_m} + \frac{v_m k_m}{k_m + G_m} \right) \right] , \tag{60}$$

$$G_a = G_m \left[1 + v_f \Big/ \left(\frac{G_m}{G_f - G_m} + \frac{v_m}{2} \right) \right] , \tag{61}$$

$$E_a = E_m v_m + E_f v_f + \frac{4 v_m v_f (\nu_m - \nu_f)^2}{(v_m/k_f) + (v_f/k_m) + (1/G_m)} , \tag{62}$$

$$\nu_a = \nu_m v_m + \nu_m v_m + \frac{v_m v_f (\nu_f - \nu_m)(\frac{1}{k_m} - \frac{1}{k_f})}{(v_m/k_f) + (v_f/k_m) + (1/G_m)} . \tag{63}$$

An excellent numerical approximation to (62) is obtained by omission of the last term. Generally, omission of the last term in (63) also yields a good approximation.

The modulus G_t has been bounded from above and below. It is here necessary, in view of future applications, to distinguish between two cases. For one composite cylinder, the G_t bounds are denoted by $G_{t(c)}^{(-)}$ and $G_{t(c)}^{(+)}$, respectively. For the fiber reinforced material regarded as a composite cylinder assemblage, the bounds are denoted by $G_t^{(-)}$ and $G_t^{(+)}$. Then,

$$G_t^{(-)} = G_m \left[1 + v_f \Big/ \left(\frac{G_m}{G_f - G_m} + \frac{(k_m + 2\, G_m)\, v_m}{2(k_m + G_m)}\right)\right], \tag{64}$$

$$G_t^{(+)} = G_{t(c)}^{(+)}. \tag{65}$$

Equation (64) is a general bound for G_t for arbitrary fiber geometry [5]. Therefore, it is also a valid lower bound for the CCM. However, since it can be shown that $G_t^{(-)} > G_{t(c)}^{(-)}$, (64) is a better lower bound than $G_{t(c)}^{(-)}$.

The expression for $G_{t(c)}^{(+)}$ is given by (138) in Section 9.

With the bounds (64) and (65) and expressions (60), (62), and (63), follow bounds for E_t and ν_t (49) and (50). It should be carefully noted that, in general,

$$E_t^{(+)} = E_t\,(G_t^{(+)}), \qquad E_t^{(-)} = E_t\,(G_t^{(-)}). \tag{66}$$

However,

$$\nu_t^{(+)} = \nu_t\,(G_t^{(-)}), \qquad \nu_t^{(-)} = \nu_t\,(G_t^{(+)}). \tag{67}$$

Here, the right sides in (66) and (67) express the functional relations defined by (49) and (50).

6. TRANSFORMATION FORMULAE

It will be recalled that the stress-strain relations (10), (11), (13), and (14) of the biaxially reinforced material hold in the x_1, x_2, x_3 system shown in Figure 1, while the stress-strain relations (42) and (43) of the uniaxially reinforced layers hold in the x_1', x_2', x_3' or x_1'', x_2'', x_3'' systems shown in Figure 2. Therefore, it is necessary to have transformation formulae for stresses, strains, and elastic moduli from one system to another.

STRESS AND STRAIN TRANSFORMATION

Define the direction cosines

$$
\begin{aligned}
\alpha_{11} &= \cos\,(x_1', x_1) = \cos\phi\,, \\
\alpha_{22} &= \cos\,(x_2', x_2) = 1\,, \\
\alpha_{33} &= \cos\,(x_3', x_3) = \cos\phi\,, \\
\alpha_{12} &= \cos\,(x_1', x_2) = 0\,, \\
\alpha_{21} &= \cos\,(x_2', x_1) = 0\,, \\
\alpha_{23} &= \cos\,(x_2', x_3) = 0\,, \\
\alpha_{32} &= \cos\,(x_3', x_2) = 0\,, \\
\alpha_{31} &= \cos\,(x_3', x_1) = -\sin\phi\,, \\
\alpha_{13} &= \cos\,(x_1', x_3) = \sin\phi\,.
\end{aligned}
\tag{68}
$$

To transform stresses σ_{ij} or strains ε_{ij} in the x_1, x_2, x_3 system into stresses σ'_{ij} or strains ε'_{ij} in the x_1', x_2', x_3' system, use the tensor transformation rules

$$
\begin{aligned}
\sigma'_{ij} &= \alpha_{ir}\,\alpha_{js}\,\sigma_{rs}, && \text{(a)} \\
\varepsilon'_{ij} &= \alpha_{ir}\,\alpha_{js}\,\varepsilon_{rs}. && \text{(b)}
\end{aligned}
\tag{69}
$$

Substituting (68) into (69), it follows that

$$\sigma'_{11} = \sigma_{11}\cos^2\phi + \sigma_{33}\sin^2\phi + 2\,\sigma_{13}\sin\phi\cos\phi, \qquad \text{(a)}$$

$$\sigma'_{33} = \sigma_{11} \sin^2 \phi + \sigma_{33} \cos^2 \phi - 2\, \sigma_{13} \sin \phi \cos \phi, \qquad \text{(b)}$$

$$\sigma'_{13} = (\sigma_{33} - \sigma_{11}) \sin \phi \cos \phi + \sigma_{13} (\cos^2 \phi - \sin^2 \phi), \qquad \text{(c)}$$

$$\sigma'_{22} = \sigma_{22}, \qquad \text{(d)}$$

$$\sigma'_{12} = \sigma_{12} \cos \phi + \sigma_{23} \sin \phi, \qquad \text{(e)}$$

$$\sigma'_{23} = - \sigma_{12} \sin \phi + \sigma_{23} \cos \phi. \qquad \text{(f)} \quad (70)$$

Obviously, the same relations hold for strain transformation.

The inverse transformation from the x'_1, x'_2, x'_3 system to the x_1, x_2, x_3 system is

$$\sigma_{11} = \sigma'_{11} \cos^2\phi + \sigma'_{33} \sin^2\phi - 2\, \sigma'_{13} \sin\phi \cos\phi, \qquad \text{(a)}$$

$$\sigma_{33} = \sigma'_{11} \sin^2\phi + \sigma'_{33} \cos^2\phi + 2\, \sigma'_{13} \sin\phi \cos\phi, \qquad \text{(b)}$$

$$\sigma_{13} = (\sigma'_{11} - \sigma'_{33}) \sin \phi \cos \phi + \sigma'_{13} (\cos^2\phi - \sin^2\phi), \qquad \text{(c)}$$

$$\sigma_{22} = \sigma'_{22}, \qquad \text{(d)}$$

$$\sigma_{12} = \sigma'_{12} \cos \phi - \sigma'_{23} \sin \phi, \qquad \text{(e)}$$

$$\sigma_{23} = \sigma'_{12} \sin \phi + \sigma'_{23} \cos \phi, \qquad \text{(f)} \quad (71)$$

with the same transformation for strains.

To obtain the transformation from the x_1, x_2, x_3 system to the x''_1, x''_2, x''_3 system and vice versa, merely replace ϕ in (70) and (71) by $(180 - \phi)$.

Then,

$$\sigma''_{11} = \sigma_{11} \cos^2 \phi + \sigma_{33} \sin^2 \phi - 2\, \sigma_{13} \sin \phi \cos \phi, \qquad \text{(a)}$$

$$\sigma''_{33} = \sigma_{11} \sin^2 \phi + \sigma_{33} \cos^2 \phi + 2\, \sigma_{13} \sin \phi \cos \phi, \qquad \text{(b)}$$

$$\sigma''_{13} = -(\sigma_{33} - \sigma_{11}) \sin \phi \cos \phi + \sigma_{13} (\cos^2 \phi - \sin^2 \phi), \qquad \text{(c)}$$

$$\sigma''_{22} = \sigma_{22}, \qquad \text{(d)}$$

$$\sigma''_{12} = - \sigma_{12} \cos \phi + \sigma_{23} \sin \phi, \qquad \text{(e)}$$

$$\sigma''_{23} = - \sigma_{12} \sin \phi - \sigma_{23} \cos \phi, \qquad \text{(f)} \quad (72)$$

$$\sigma_{11} = \sigma''_{11} \cos^2\phi + \sigma''_{33} \sin^2\phi + 2\,\sigma''_{13} \sin\phi \cos\phi\,, \quad \text{(a)}$$

$$\sigma_{33} = \sigma''_{11} \sin^2\phi + \sigma''_{33} \cos^2\phi - 2\,\sigma''_{13} \sin\phi \cos\phi\,, \quad \text{(b)}$$

$$\sigma_{13} = -(\sigma''_{11} - \sigma''_{33}) \sin\phi \cos\phi + \sigma''_{13}(\cos^2\phi - \sin^2\phi), \quad \text{(c)}$$

$$\sigma_{22} = \sigma''_{22}\,, \quad \text{(d)}$$

$$\sigma_{12} = -\,\sigma''_{12} \cos\phi - \sigma''_{23} \sin\phi, \quad \text{(e)}$$

$$\sigma_{23} = \sigma''_{12} \sin\phi - \sigma''_{23} \cos\phi\,, \quad \text{(f)} \quad (73)$$

with similar expressions for strain transformation.

ELASTIC MODULI TRANSFORMATION

In view of equations (23) and (28), it is necessary to transform the elastic moduli of the transversely isotropic layers from the x'_1, x'_2, x'_3 and x''_1, x''_2, x''_3 systems to the x_1, x_2, x_3 system.

Denote the elastic moduli in (42) and (43) by C'_{ijkl}. These are the moduli of a transversely isotropic layer with respect to its own (primed) system of coordinates, which is the only coordinate system which brings forth the transverse symmetry. Similarly, the moduli of the layer whose fibers are at (180 - ϕ) orientation may be denoted as C''_{ijkl}.

It is obvious that

$$C'_{ijkl} = C''_{ijkl}, \quad (74)$$

since the layers are made of the same material.

Let the C'_{ijkl} moduli, transformed to the x_1, x_2, x_3 system be denoted by ${}'C_{ijkl}$. Then,

$${}'C_{ijkl} = \alpha'_{pi}\, \alpha'_{qj}\, \alpha'_{rk}\, \alpha'_{sl}\, C'_{pqrs}, \quad (75)$$

where momentarily α'_{ij} are the direction cosines (68). Similarly, transform C''_{ijkl} to the x_1, x_2, x_3 system and denote the result by ${}''C_{ijkl}$. Then

$${}''C_{ijkl} = \alpha''_{pi}\, \alpha''_{qj}\, \alpha''_{rk}\, \alpha''_{sl}\, C'_{pqrs}\,. \quad (76)$$

Here, α''_{ij} are the direction cosines (68) with ϕ replaced by (180 - ϕ) and (74) has been used. The transformed moduli ${}'C_{ijkl}$ and ${}''C_{ijkl}$ enter into (23).

In view of (42) and (43), the C'_{ijkl} have the following nonvanishing components:

$$C'_{1111} = C'_{11} ,$$

$$C'_{2222} = C'_{3333} = C'_{22} ,$$

$$C'_{1122} = C'_{2211} = C'_{1133} = C'_{3311} = C'_{12},$$

$$C'_{2233} = C'_{3322} = C'_{23},$$

$$C'_{1212} = C'_{2121} = C'_{1221} = C'_{2112} = C'_{44},$$

$$C'_{1313} = C'_{3131} = C'_{1331} = C'_{3113} = C'_{44},$$

$$C'_{2323} = C'_{3232} = C'_{2332} = C'_{3223} = \frac{1}{2} (C'_{22} - C'_{23}). \tag{77}$$

Considering now the transformation laws (75), (76) with (68), the following features may be observed: Whenever subscripts 1, 2, 3, in $'C_{ijkl}$ and $''C_{ijkl}$ appear only even numbers of times, the transformation laws (75) and (76) involve only even powers of sin ϕ and cos ϕ or of sin (180 - ϕ). Therefore, in such cases, both transformation laws give the same results. Thus, the following relations result:

$$'C_{1111} = ''C_{1111} , \qquad 'C_{1212} = ''C_{1212} ,$$

$$'C_{2222} = ''C_{2222} , \qquad 'C_{2323} = ''C_{2323} ,$$

$$'C_{3333} = ''C_{3333} , \qquad 'C_{3131} = ''C_{3131} ,$$

$$'C_{1122} = ''C_{1122} ,$$

$$'C_{2233} = ''C_{2233},$$

$$'C_{3311} = ''C_{3311}. \tag{78}$$

On the other hand, in all the remaining $'C_{ijkl}$ and $''C_{ijkl}$, cos ϕ and cos (180 - ϕ) appear only in odd powers. Therefore, in this case,

$$'C_{ijkl} = - ''C_{ijkl}$$

and, therefore, these components do not enter into (23). Consequently, the elastic moduli tensor

$$\tilde{C}_{ijkl} = \frac{1}{2} ({}'C_{ijkl} + {}''C_{ijkl}) \tag{79}$$

has only the components (78). Accordingly, $\tilde{C}_{ijkl}$ is orthotropic.

Obviously, completely analogous reasoning applies for the compliances. Accordingly,

$$\tilde{S}_{ijkl} = \frac{1}{2} ({}'S_{ijkl} + {}''S_{ijkl}) \tag{80}$$

is orthotropic.

The components of $\tilde{C}_{ijkl}$ will now be given explicitly in terms of C'_{ijkl} on the basis of (75) and (68). Six-by-six matrix notation will again be used; i.e.,

$$\begin{aligned} &\tilde{C}_{1111} = \tilde{C}_{11}, \qquad \tilde{C}_{1122} = \tilde{C}_{12}, \text{ etc.,} \\ &\tilde{C}_{1212} = \tilde{C}_{44}, \qquad \tilde{C}_{2323} = \tilde{C}_{55}, \qquad \tilde{C}_{3131} = \tilde{C}_{66}. \end{aligned} \tag{81}$$

A long and tedious calculation yields

$$\begin{aligned} \tilde{C}_{11} = \cos^4\phi\, C'_{11} + \sin^4\phi\, C'_{22} + 2 \sin^2\phi \cos^2\phi\, C'_{12} \\ + 4 \sin^2\phi \cos^2\phi\, C'_{44}, \end{aligned} \tag{a}$$

$$\tilde{C}_{22} = C'_{22}\,. \tag{b}$$

$$\begin{aligned} \tilde{C}_{33} = \sin^4\phi\, C'_{11} + \cos^4\phi\, C'_{22} + 2 \sin^2\phi \cos^2\phi\, C'_{12} \\ + 4 \sin^2\phi \cos^2\phi\, C'_{44}, \end{aligned} \tag{c}$$

$$\tilde{C}_{12} = \cos^2\phi\, C'_{12} + \sin^2\phi\, C'_{23}, \tag{d}$$

$$\tilde{C}_{23} = \sin^2\phi\, C'_{12} + \cos^2\phi\, C'_{23}, \tag{e}$$

$$\begin{aligned} C'_{13} = \cos^2\phi \sin^2\phi\, C'_{11} + \cos^2\phi \sin^2\phi\, C'_{22} \\ + (\sin^4\phi + \cos^4\phi)\, C'_{12} - 4 \sin^2\phi \cos^2\phi\, C'_{44}, \end{aligned} \tag{f}$$

$$\tilde{C}_{44} = \cos^2\phi\, C'_{44} + \sin^2\phi\, \frac{1}{2} (C'_{22} - C'_{23}), \tag{g}$$

$$\tilde{C}_{55} = \sin^2\phi \, C'_{44} + \cos^2\phi \, \frac{1}{2} \, (C'_{22} - C'_{23}), \qquad \text{(h)}$$

$$\tilde{C}_{66} = \cos^2\phi \, \sin^2\phi \, (C'_{11} + C'_{12} - 2\, C'_{12}) + (\sin^2\phi - \cos^2\phi)^2 \, C'_{44} \, . \qquad \text{(i)} \quad (82)$$

It should be remembered that the primed moduli on the right-hand sides are defined by (42) and (43).

Similar results can be written for compliances.

7. BOUNDING FOR LAYER MODEL (LM)

The fundamental relations for bounding of effective elastic moduli and compliances on the basis of the LM are (23) and (28), which yield

$$(\tilde{C}_{ijkl} - C^*_{ijkl})\, \varepsilon^o_{ij}\, \varepsilon^o_{kl} \geq 0 \;, \tag{83}$$

$$(\tilde{S}_{ijkl} - S^*_{ijkl})\, \sigma^o_{ij}\, \sigma^o_{kl} \geq 0 \;. \tag{84}$$

Here, $\tilde{C}_{ijkl}$ and S_{ijkl} are known matrices which are given in terms of the known single-layer moduli by the transformation (82), whereas C^*_{ijkl} and S^*_{ijkl} are the unknown composite moduli and compliance matrices. Thus, (83) and (84) are equivalent to the statement that the matrices $\tilde{C}_{ijkl} - C^*_{ijkl}$ and $\tilde{S}_{ijkl} - S^*_{ijkl}$ are positive definite.

It will now be convenient to switch to six-by-six matrix notation. For moduli and compliances, the six-by-six notation is defined by (81). The strains and stresses are redefined according to the scheme

$$\begin{aligned} &\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3 \;, \\ &2\varepsilon_{12} = \varepsilon_4, \quad 2\varepsilon_{23} = \varepsilon_5, \quad 2\varepsilon_{31} = \varepsilon_6 \;, \end{aligned} \tag{85}$$

$$\begin{aligned} &\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3 \;, \\ &\sigma_{12} = \sigma_4, \quad \sigma_{23} = \sigma_5, \quad \sigma_{31} = \sigma_6 \;. \end{aligned} \tag{86}$$

Also, denote for convenience

$$\Delta C_{ijkl} = \tilde{C}_{ijkl} - C^*_{ijkl} \;, \qquad \text{(a)}$$

$$\Delta S_{ijkl} = \tilde{S}_{ijkl} - S^*_{ijkl} \;. \qquad \text{(b)} \qquad (87)$$

It should be remembered that all the matrices in (87) are orthotropic. Thus, these matrices written out in detail in six-by-six matrix notation are

$$
\Delta \underline{C} = \begin{bmatrix} \Delta C_{11} & \Delta C_{12} & \Delta C_{13} & 0 & 0 & 0 \\ \Delta C_{12} & \Delta C_{22} & \Delta C_{23} & 0 & 0 & 0 \\ \Delta C_{13} & \Delta C_{23} & \Delta C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta C_{66} \end{bmatrix} , \tag{88}
$$

$$
\Delta \underline{S} = \begin{bmatrix} \Delta S_{11} & \Delta S_{12} & \Delta S_{13} & 0 & 0 & 0 \\ \Delta S_{12} & \Delta S_{22} & \Delta S_{23} & 0 & 0 & 0 \\ \Delta S_{13} & \Delta S_{23} & \Delta S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta S_{66} \end{bmatrix} . \tag{89}
$$

The matrices (88) and (89) must be positive definite according to (83) and (84). It is seen that these matrices are completely defined by the submatrices.

$$
\Delta \underline{C}^1 = \begin{bmatrix} \Delta C_{11} & \Delta C_{12} & \Delta C_{13} \\ \Delta C_{12} & \Delta C_{22} & \Delta C_{23} \\ \Delta C_{13} & \Delta C_{23} & \Delta C_{33} \end{bmatrix} , \tag{90}
$$

$$
\Delta \underline{C}^2 = \begin{bmatrix} \Delta C_{44} & 0 & 0 \\ 0 & \Delta C_{55} & 0 \\ 0 & 0 & \Delta C_{66} \end{bmatrix} , \tag{91}
$$

$$\Delta\underline{S}^1 = \begin{bmatrix} \Delta S_{11} & \Delta S_{12} & \Delta S_{13} \\ \Delta S_{12} & \Delta S_{22} & \Delta S_{23} \\ \Delta S_{13} & \Delta S_{23} & \Delta S_{33} \end{bmatrix} , \tag{92}$$

$$\Delta\underline{S}^2 = \begin{bmatrix} \Delta S_{44} & 0 & 0 \\ 0 & \Delta S_{55} & 0 \\ 0 & 0 & \Delta S_{66} \end{bmatrix} . \tag{93}$$

For $\Delta\underline{C}$ to be positive definite, it is necessary and sufficient that $\Delta\underline{C}^1$ and $\Delta\underline{C}^2$ be positive definite. Similarly, for $\Delta\underline{S}$ to be positive definite, it is necessary and sufficient that $\Delta\underline{S}^1$ and $\Delta\underline{S}^2$ be positive definite.

As is well known, a necessary and sufficient condition for a matrix to be positive definite is that its eigenvalues be positive. Since the matrices $\Delta\underline{C}^2$ and $\Delta\underline{S}^2$ are already in diagonal form, as can be seen from (91) and (93), the required positive definiteness of these matrices is simply expressed by the conditions

$$\Delta C_{44} \geq 0 , \qquad \text{(a)}$$
$$\Delta C_{55} \geq 0 , \qquad \text{(b)}$$
$$\Delta C_{66} \geq 0 . \qquad \text{(c)} \quad (94)$$

$$\Delta S_{44} \geq 0 , \qquad \text{(a)}$$
$$\Delta S_{55} \geq 0 , \qquad \text{(b)}$$
$$\Delta S_{66} \geq 0 . \qquad \text{(c)} \quad (95)$$

In view of (87 a), in six-by-six notation, (94) can be written in the form

$$C^*_{44} \leq \tilde{C}_{44} , \qquad \text{(a)}$$
$$C^*_{55} \leq \tilde{C}_{55} , \qquad \text{(b)}$$
$$C^*_{66} \leq \tilde{C}_{66} , \qquad \text{(c)} \quad (96)$$

which provides upper bounds for the shear moduli of the biaxially reinforced material.

In view of (87 b), in six-by-six matrix notation, (95) assumes the form

$$S^{*}_{44} \leq \tilde{S}_{44} \;, \qquad \text{(a)}$$

$$S^{*}_{55} \leq \tilde{S}_{55} \;, \qquad \text{(b)}$$

$$S^{*}_{66} \leq \tilde{S}_{66} \;. \qquad \text{(c)} \qquad (97)$$

Now recall that since C^{*}_{ijkl} and S^{*}_{ijkl} are effective elastic moduli and effective compliances of the biaxially reinforced material, they are reciprocal to one another. It should also be remembered that because of orthotropy, each of these matrices is of the form (88) and (89); (compare also (10) to (14)). Accordingly, the submatrix $\underline{C}^{*2}$, given by

$$\underline{C}^{*2} = \begin{bmatrix} C^{*}_{44} & 0 & 0 \\ 0 & C^{*}_{55} & 0 \\ 0 & 0 & C^{*}_{66} \end{bmatrix} , \qquad (98)$$

is the reciprocal of the submatrix $\underline{S}^{*2}$, given by

$$\underline{S}^{*2} = \begin{bmatrix} S^{*}_{44} & 0 & 0 \\ 0 & S^{*}_{55} & 0 \\ 0 & 0 & S^{*}_{66} \end{bmatrix} . \qquad (99)$$

Since these matrices are diagonal, their reciprocity is expressed by the relations

$$S^{*}_{44} = \frac{1}{C^{*}_{44}} \;, \qquad \text{(a)}$$

$$S^*_{55} = \frac{1}{C^*_{55}} , \qquad \text{(b)}$$

$$S^*_{66} = \frac{1}{C^*_{66}} . \qquad \text{(c)} \qquad (100)$$

It is seen that when (100) is introduced into (97), lower bounds for C^*_{44}, C^*_{55}, and C^*_{66} are obtained, while upper bounds are given by (96). We shall now adopt the physically more meaningful notation

$$G^*_{12} = C^*_{44} , \qquad \text{(a)}$$

$$G^*_{23} = C^*_{55} , \qquad \text{(b)}$$

$$G^*_{31} = C^*_{66} . \qquad \text{(c)} \qquad (101)$$

These are the three different shear moduli of the biaxially reinforced material. The subscripts indicate the directions of shear. Then (96), (97), and (100) provide the following bounds for the shear moduli:

$$\frac{1}{\tilde{S}_{44}} \leq G^*_{12} \leq \tilde{C}_{44} , \qquad \text{(a)}$$

$$\frac{1}{\tilde{S}_{55}} < G^*_{23} \leq \tilde{C}_{55} , \qquad \text{(b)}$$

$$\frac{1}{\tilde{S}_{66}} \leq G^*_{31} \leq \tilde{C}_{66} . \qquad \text{(c)} \qquad (102)$$

It will be recalled that the extreme right and left sides in the inequalities (102) are expressed in terms of the single-layer moduli and compliances and the reinforcement angle ϕ, by relations of type (82). Therefore, (102) provides lower and upper bounds for the effective shear moduli of the biaxially stiffened material. Such bounds will be given in detail in the next section.

Unfortunately, the situation for the other six effective moduli and compliances is not so simple, since the matrices (90) and (92) are not diagonal. Furthermore, the effective moduli and compliances entering into (90) and (92) through (87) are now contained in the submatrices

$$\underline{C}^{*1} = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{13} \\ C^*_{12} & C^*_{22} & C^*_{23} \\ C^*_{13} & C^*_{23} & C^*_{33} \end{bmatrix} , \tag{103}$$

$$\underline{S}^{*1} = \begin{bmatrix} S^*_{11} & S^*_{12} & S^*_{13} \\ S^*_{12} & S^*_{22} & S^*_{23} \\ S^*_{13} & S^*_{23} & S^*_{33} \end{bmatrix} . \tag{104}$$

Because of the reciprocity of C^*_{ijkl} and S^*_{ijkl}, the submatrices (103) and (104) are reciprocal. However, the reciprocity relations are now of a very complicated nature, unlike the simple relations (100).

The mathematical problem with which we are now confronted is as follows: Given that the matrices

$$\Delta\underline{C}^1 = \underline{\tilde{C}}^1 - \underline{C}^{*1}, \tag{105}$$

$$\Delta\underline{S}^1 = \underline{\tilde{S}}^1 - \underline{S}^{*1}, \tag{106}$$

are positive definite, where the matrices on the right sides are of the forms (103) and (104), and furthermore each of the matrices on the right sides of (105) and (106) is by itself positive definite, and $\underline{C}^{*1}$ and $\underline{S}^{*1}$ are reciprocal, find upper and lower bounds on the components of $\underline{C}^{*1}$.

A solution to such a problem does not seem to have appeared in the mathematical literature. The present bounding method will be used only for shear moduli bounding, and a different method will be adopted for the other moduli and compliances.

8. BOUNDS FOR SHEAR MODULI (LM AND CCM)

The right sides of (102) are given by (82 h,i,j). These are now rewritten with the aid of (44) to (48). From (45), (46), and (82 h,i), we have

$$\tilde{C}_{44} = \cos^2\phi\ G_a + \sin^2\phi\ G_t, \tag{107}$$

$$\tilde{C}_{55} = \sin^2\phi\ G_a + \cos^2\phi\ G_t. \tag{108}$$

From (44) to (48) and (82 j), we have

$$\tilde{C}_{66} = \frac{1}{4} \sin^2 2\phi\ [E_a + G_t + K_t\ (1 - 2\nu_a)^2] + \cos^2 2\phi\ G_a. \tag{109}$$

These expressions define the upper bounds in (102).

To obtain lower bounds, the compliance analogues of (82 h,i,j) are needed. But here great caution must be exercised in the interpretation of the six-by-six matrix notation. Note that a pure shear stress-strain relation for orthotropic material is, for example,

$$\sigma_{12} = 2C_{12}\ \varepsilon_{12}\ , \tag{110}$$

which becomes

$$\sigma_4 = C_{44}\ \varepsilon_4\ , \tag{111}$$

in the notation (81), (85), and (86). The inverse is

$$\varepsilon_4 = S_{44}\ \sigma_4\ , \tag{112}$$

and thus,

$$S_{44} = \frac{1}{C_{44}}\ . \tag{113}$$

However, in tensor notation,

$$\varepsilon_{12} = 2S_{1212}\ \sigma_{12}\ , \tag{114}$$

whereas (112) can be written as

$$\varepsilon_{12} = \frac{1}{2} S_{44}\ \sigma_{12}\ . \tag{115}$$

Thus, from (114) and (115),

$$S_{44} = 4S_{1212}, \tag{116}$$

and similarly,

$$S_{55} = 4S_{2323}, \quad S_{66} = 4S_{3131}, \tag{117}$$

in contrast to relations of type (81 b). On the other hand, relations of type (81 a) remain of the same form for compliances.

In veiw of the foregoing comments and the fact that (82) is really fourth-rank tensor transformation, (82 h,i,j) assume the following form for six-by-six compliance matrix components:

$$\tilde{S}_{44} = \cos^2\phi \; S'_{44} + \sin^2\phi \; 2(S'_{22} - S'_{23}), \quad \text{(a)}$$

$$\tilde{S}_{55} = \sin^2\phi \; S'_{44} + \cos^2\phi \; 2(S'_{22} - S'_{23}), \quad \text{(b)}$$

$$\tilde{S}_{66} = 4\cos^2\phi \sin^2\phi \; (S'_{11} + S'_{22} - 2S'_{12}) + (\sin^2\phi - \cos^2\phi)^2 \; S'_{44} \, . \quad \text{(c)} \tag{118}$$

The compliances in (118) are defined by the inversion of (42) and (43), which may be written as

$$\bar{\varepsilon}'_{11} = S'_{11}\,\bar{\sigma}'_{11} + S'_{12}\,\bar{\sigma}'_{22} + S'_{12}\,\bar{\sigma}'_{33} \, , \quad \text{(a)}$$

$$\bar{\varepsilon}'_{22} = S'_{12}\,\bar{\sigma}'_{11} + S'_{22}\,\bar{\sigma}'_{22} + S'_{23}\,\bar{\sigma}'_{33} \, , \quad \text{(b)}$$

$$\bar{\varepsilon}'_{33} = S'_{12}\,\bar{\sigma}'_{11} + S'_{23}\,\bar{\sigma}'_{22} + S'_{22}\,\bar{\sigma}'_{33} \, . \quad \text{(c)} \tag{119}$$

$$2\bar{\varepsilon}'_{12} = S'_{44}\,\bar{\sigma}'_{12} \, , \quad \text{(a)}$$

$$2\bar{\varepsilon}'_{23} = \frac{1}{2}(S'_{22} - S'_{23})\,\sigma'_{23} \, , \quad \text{(b)}$$

$$2\varepsilon'_{31} = S'_{44}\,\sigma'_{31} \, . \quad \text{(c)} \tag{120}$$

Thus, from (120) and (43),

$$S'_{44} = \frac{1}{C'_{44}} = \frac{1}{G_a}, \tag{121}$$

$$S'_{22} - S'_{23} = \frac{1}{C'_{22} - C'_{23}} = \frac{1}{2G_t} . \tag{122}$$

By inserting (121) and (122) into (118 a,b), we find

$$\tilde{S}_{44} = \frac{\cos^2\phi}{G_a} + \frac{\sin^2\phi}{G_t}, \tag{123}$$

$$\tilde{S}_{55} = \frac{\sin^2\phi}{G_a} + \frac{\cos^2\phi}{G_t}. \tag{124}$$

By combining (123), (124), (107), and (108) with (102 a,b), we find the following pair of simple bounds:

$$\left[\frac{\cos^2\phi}{G_a} + \frac{\sin^2\phi}{G_t}\right]^{-1} \leq G^*_{12} \leq \cos^2\phi\, G_a + \sin^2\phi\, G_t , \tag{125}$$

$$\left[\frac{\sin^2\phi}{G_a} + \frac{\cos^2\phi}{G_t}\right]^{-1} \leq G^*_{23} \leq \sin^2\phi\, G_a + \cos^2\phi\, G_t . \tag{126}$$

In order to avoid very heavy algebra, a different procedure will be adopted to derive an expression for a lower bound for G^*_{13}. For this purpose, (119) will be rewritten in terms of Young's moduli and Poisson's ratio defined in Section 5. Thus,

$$\bar{\varepsilon}'_{11} = \frac{\bar{\sigma}'_{11}}{E_a} - \frac{\nu_a}{E_t}\bar{\sigma}'_{22} - \frac{\nu_a}{E_t}\bar{\sigma}'_{33} , \tag{a}$$

$$\bar{\varepsilon}'_{22} = -\frac{\nu_a}{E_t}\bar{\sigma}'_{11} + \frac{\bar{\sigma}'_{22}}{E_t} - \frac{\nu_t}{E_t}\bar{\sigma}'_{33}, \tag{b}$$

$$\bar{\epsilon}'_{33} = -\frac{\nu_a}{E_t}\bar{\sigma}'_{11} - \frac{\nu_t}{E_t}\bar{\sigma}'_{22} + \frac{\bar{\sigma}'_{33}}{E_t} \quad . \qquad \text{(c)} \qquad (127)$$

It is seen that (127) is merely the specialization of (13) with (15) and (16) to a transversely isotropic material. Comparing (127) with (119), we can write

$$S'_{11} = \frac{1}{E_a} \,, \qquad \text{(a)}$$

$$S'_{22} = \frac{1}{E_t} \,, \qquad \text{(b)}$$

$$S'_{12} = -\frac{\nu_a}{E_t} \,, \qquad \text{(c)}$$

$$S'_{23} = -\frac{\nu_t}{E_t} \,. \qquad \text{(d)} \qquad (128)$$

Using (121) and (128) in (118), we find

$$\tilde{S}_{66} = \sin^2 2\phi \; \frac{1}{E_a} + \frac{1 + 2\nu_a}{E_t} + \cos^2 2\phi \, \frac{1}{G_a} \,. \qquad (129)$$

By combining (109) and (129) with (102 c) we finally obtain bounds on G^*_{31}. Thus

$$\left[\sin^2 2\phi \left(\frac{1}{E_a} + \frac{1 + 2\nu_a}{E_t}\right) + \frac{\cos^2 2\phi}{G_a}\right]^{-1} \leq G^*_{31} \leq \sin^2 2\phi \, \frac{1}{4} \, [E_a + G_t + K_t(1-2\nu_a)^2]$$

$$+ \cos^2 2\phi \, G_a \quad . \qquad (130)$$

Equations (125), (126), and (130) thus provide lower and upper bounds for the three shear moduli of a biaxially fiber reinforced material in terms of the transversely isotropic layer moduli.

Note that when $\phi = 0$, the biaxially reinforced material degenerates into a uniaxially stiffened material with fibers in the x_1 direction (Figure 2). In this event, (125), (126), and (130) reduce to

$$G^*_{12} = G_a,$$

$$G^*_{23} = G_t,$$

$$G^*_{31} = G_a \,, \qquad (131)$$

as they should.

On the other hand, when $\phi = 90°$, the biaxially reinforced material degenerates into a uniaxially stiffened material with fibers in the x_3 direction. In this event, (125), (126), and (130) reduce to

$$G_{12}^* = G_t,$$

$$G_{23}^* = G_a,$$

$$G_{31}^* = G_a, \tag{132}$$

as they should.

We now consider the situation when the layers are described by the composite cylinder model (CCM), Figure 3b. The bounding method to be used has been described in general terms in Section 4. Fortunately, it is not necessary to give any further analysis based on this method, for it turns out that an entirely equivalent procedure is to take for elastic moduli of the layers those predicted by Hashin and Rosen [1], and to use these in the LM bounding method.

The truth of this statement follows from the method of analysis employed in [1]. If average shear strains or shear stresses are applied to a uniaxial fiber composite which is described by the CCM, then the strain or stress energies stored in it are given by the usual expressions for strain or stress energies in homogeneous materials, with the effective shear moduli replacing the homogeneous shear moduli. Therefore, the LM bounding method which was based on a homogeneous layer becomes immediately applicable for CCM layers, with values predicted by the CCM method used for the layer moduli.

Turning first to (125) and (126) it is important to note that the CCM method gives a closed expression for G_a but only lower and upper bounds for G_t. Denoting such bounds on G_t by $G_t^{(-)}$ and $G_t^{(+)}$, we note the following relations:

$$\cos^2\phi\, G_a + \sin^2\phi\, G_t \le \cos^2\phi\, G_a + \sin^2\phi\, G_t^{(+)}, \qquad \text{(a)}$$

$$\left[\frac{\cos^2\phi}{G_a} + \frac{\sin^2\phi}{G_t^{(-)}}\right]^{-1} \le \left[\frac{\cos^2\phi}{G_a} + \frac{\sin^2\phi}{G_t}\right]^{-1}, \qquad \text{(b)}$$

$$\sin^2\phi\, G_a + \cos^2\phi\, G_t \le \sin^2\phi\, G_a + \cos^2\phi\, G_t^{(+)}, \qquad \text{(c)}$$

$$\left[\frac{\sin^2\phi}{G_a} + \frac{\cos^2\phi}{G_t^{(-)}}\right]^{-1} \le \left[\frac{\sin^2\phi}{G_a} + \frac{\cos^2\phi}{G_t}\right]^{-1}. \qquad \text{(d)} \tag{133}$$

The best bounds available for G_t of the CCM are given by (64) and (65). Thus, with the use of these in (133), the bounds (125) and (126) assume the form

$$\left[\frac{\cos^2\phi}{G_a} + \frac{\sin^2\phi}{G_t^{(-)}}\right]^{-1} \leq G_{12}^* \leq \cos^2\phi\ G_a + \sin^2\phi\ G_{t(c)}^{(+)} \quad , \tag{134}$$

$$\left[\frac{\sin^2\phi}{G_a} + \frac{\cos^2\phi}{G_t^{(-)}}\right]^{-1} \leq G_{23}^* \leq \sin^2\phi\ G_a + \cos^2\phi\ G_{t(c)}^{(+)} \quad . \tag{135}$$

Similar remarks concern (130). G_t appears again on the right side, while all other moduli on this side have closed expressions on the basis of the CCM. On the left side, all the moduli except E_t have closed expressions. Bounds on E_t can be expressed in terms of bounds on G_t by (49). Note in this respect (62). Let

$$E_t^{(-)} = E_t[G_t^{(-)}], \tag{136}$$

where the functional relationship in (136) is given by (49) with inequality arguments similar to those used in the derivation of (134) and (135). We find on the basis of (130),

$$\left[\sin^2 2\phi\left(\frac{1}{E_a} + \frac{1 + 2\nu_a}{E_t^{(-)}}\right) + \frac{\cos^2 2\phi}{G_a}\right]^{-1} \leq G_{31}^*$$

$$\leq \frac{1}{4}\sin^2 2\phi\left[E_a + G_{t(c)}^{(+)} + K_t(1 - 2\nu_a)^2\right] + \cos^2 2\phi\ G_a \ . \tag{137}$$

9. NUMERICAL RESULTS FOR SHEAR MODULI BOUNDS

By utilizing the theoretical results derived in Section 8, bounds on the effective shear moduli will be calculated in the following.

Expressions for bounds are given by (134), (135), and (137):

$$\left[\frac{\cos^2\phi}{G_a} + \frac{\sin^2\phi}{G_t^{(-)}}\right]^{-1} \leq G_{12}^* \leq \cos^2\phi\ G_a + \sin^2\phi\ G_{t(c)}^{(+)} \tag{134}$$

$$\left[\frac{\sin^2\phi}{G_a} + \frac{\cos^2\phi}{G_t^{(-)}}\right]^{-1} \leq G_{23}^* \leq \sin^2\phi\ G_a + \cos^2\phi\ G_{t(c)}^{(+)} \tag{135}$$

$$\left[\sin^2 2\phi\ \left(\frac{1}{E_a} + \frac{1+2\nu a}{E_t^{(-)}}\right) + \frac{\cos^2 2\phi}{G_a}\right]^{-1} \leq G_{31}^*$$

$$\leq \frac{1}{4}\sin^2 2\phi\ [E_a + G_{t(c)}^{(+)} + K_t\ (1-2\nu a)^2] + \cos^2 2\phi\ G_a. \tag{137}$$

The moduli appearing in these equations are calculated from expressions given in Section 5.

$$E_a = E_m v_m + E_f v + \frac{4v_m v_f\ (\nu_m - \nu_f)^2}{\left(v_m/k_f\right)+\left(v_f/k_m\right)+\left(1/G_m\right)}, \tag{62}$$

$$\nu_a = \nu_m v_m + \nu_f v_f + \frac{v_m v_f (\nu_f - \nu_m)\left(\frac{1}{k_m} - \frac{1}{k_f}\right)}{\left(v_m/k_f\right)+\left(v_f/k_m\right)+(1/G_m)}, \tag{63}$$

$$K_t = k_m\left[1 + v_f \Big/ \left(\frac{k_m}{k_f-k_m} + \frac{v_m k_m}{k_m + G_m}\right)\right], \tag{60}$$

$$G_a = G_m\left[1 + v_f \Big/ \left(\frac{G_m}{G_f-G_m} + \frac{1}{2}v_m\right)\right], \tag{61}$$

$$G_t^{(-)} = G_m\left[1 + v_f \Big/ \left(\frac{G_m}{G_f-G_m} + \frac{v_m(k_m + 2G_m)}{2(k_m + G_m)}\right)\right]. \tag{64}$$

$$G_{t(c)}^{(+)} = G_m \left[1 + \frac{2\left(1 - \nu_m\right)}{1 - 2\nu_m} v_f \bar{A}_4^{\varepsilon} \right] , \qquad (138)$$

In (137), $E_t^{(-)}$ is obtained from (49 a), replacing G_t by $G_t^{(-)}$.

$$E_t^{(-)} = \frac{4G_t^{(-)} K_t}{K_t + \psi G_t^{(-)}} , \qquad (139)$$

$$\psi = 1 + \frac{4K_t \nu_a^2}{E_a} . \qquad (49\ b)$$

The elastic constants both for the fiber and the matrix materials (denoted by the subscripts f and m, respectively) are given by

$$G = \frac{E}{2(1 + \nu)} , \qquad (a)$$

$$\lambda = \frac{\nu E}{(1 + \nu)(1-2\nu)} , \qquad (b)$$

$$k = \lambda + G . \qquad (c) \qquad (140)$$

An expression for $G_{t(c)}^{(+)}$ is given in [1]. $\bar{A}_4^{\varepsilon}$ can be obtained from the solution of the following six linear algebraic equations, [1]:

$$
\begin{bmatrix}
1 & 1/v_f & v_f^2 & v_f & 0 & 0 \\
0 & -\frac{3-4\nu_m}{3-2\nu_m} - \frac{1}{f} & -2v_f & \frac{v_f}{1-2\nu_m} & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 \\
0 & -\frac{3-4\nu_m}{3-2\nu_m} & -2 & \frac{1}{1-2\nu_m} & 0 & \frac{3-4\nu_f}{3-2\nu_f} \\
1 & \frac{3}{3-2\nu_m} & -3 & \frac{1}{1-2\nu_m} & -\eta & \frac{3\eta}{3-2\nu_f} \\
0 & -\frac{1}{3-2\nu_m} & 2 & -\frac{1}{1-2\nu_m} & 0 & \frac{\eta}{3-2\nu_f}
\end{bmatrix}
\begin{Bmatrix} \bar{A}_1^{\varepsilon} \\ \bar{A}_2^{\varepsilon} \\ \bar{A}_3^{\varepsilon} \\ \bar{A}_4^{\varepsilon} \\ \bar{A}_5^{\varepsilon} \\ \bar{A}_6^{\varepsilon} \end{Bmatrix}
=
\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix},
\tag{141}
$$

where

$$
\eta = G_f/G_m \quad . \tag{142}
$$

The results given in this section have been programmed for the IBM 7094 digital computer. For composite materials whose properties are listed below, sample calculations have been performed.

Fiber glass/Epoxy

$E_f = 10.5 \times 10^6$ psi $\qquad E_m = 0.5 \times 10^6$ psi

$\nu_f = 0.2 \qquad \nu_m = 0.3 \times 10^6$

Boron/Magnesium

$E_f = 60 \times 10^6$ psi $\qquad E_m = 6.5 \times 10^6$ psi

$\nu_f = 0.3 \qquad \nu_m = 0.3$

The results are shown in graphical form in Figures 4 through 39. Figures 4 through 21 are for fiber glass/epoxy; and Figures 22 through 39 for boron/magnesium. Shown are variations of shear moduli with angle of reinforcement ϕ for fixed volume fraction and variations with fiber volume fractions for fixed angle of reinforcement ϕ.

It is seen that the bounds on G^*_{12} and G^*_{23} are extremely close, while those on G^*_{13} are further apart.

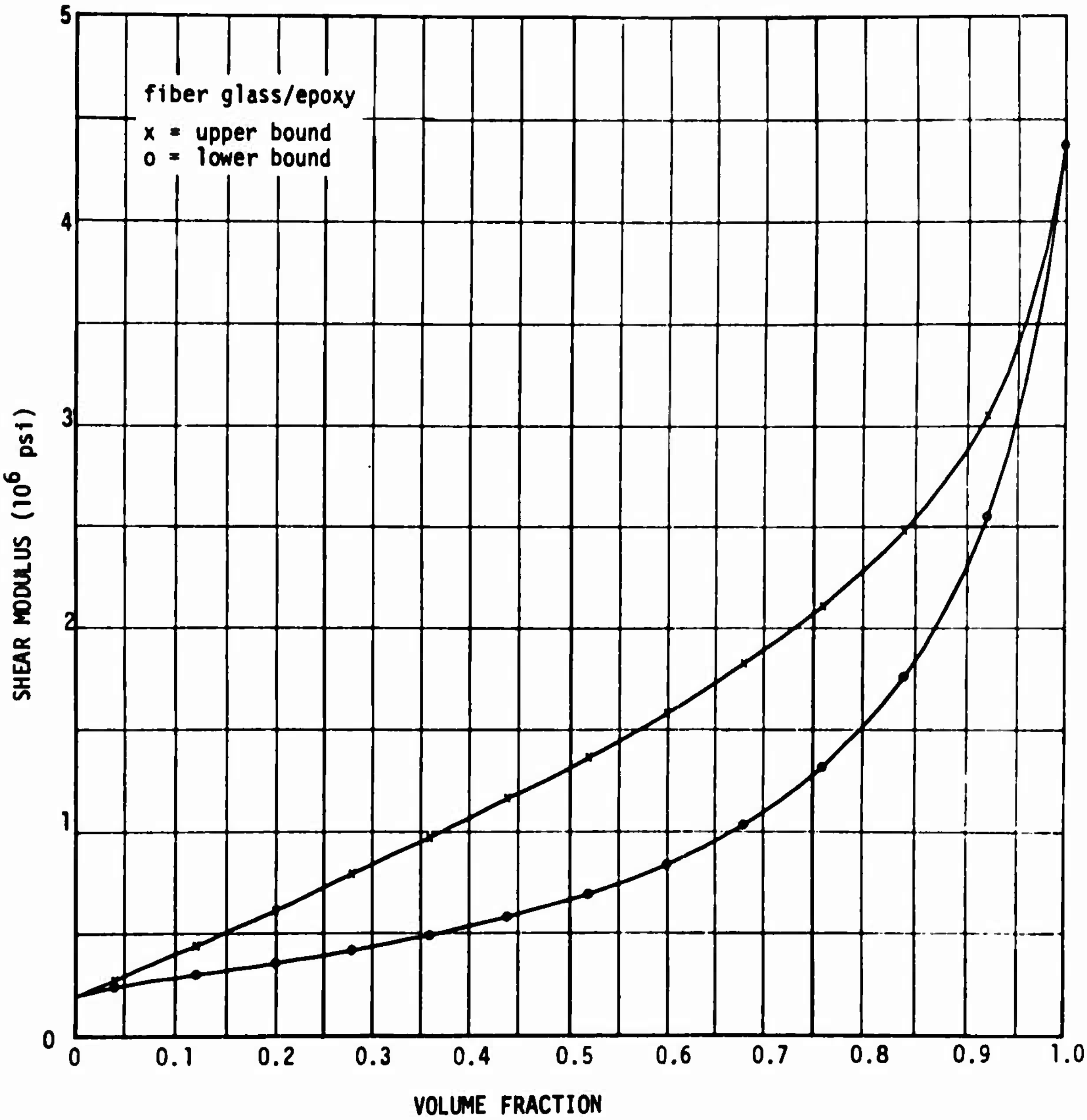

Figure 4. Shear Modulus G_{31}^* vs. Volume Fraction
ϕ = 30 degrees

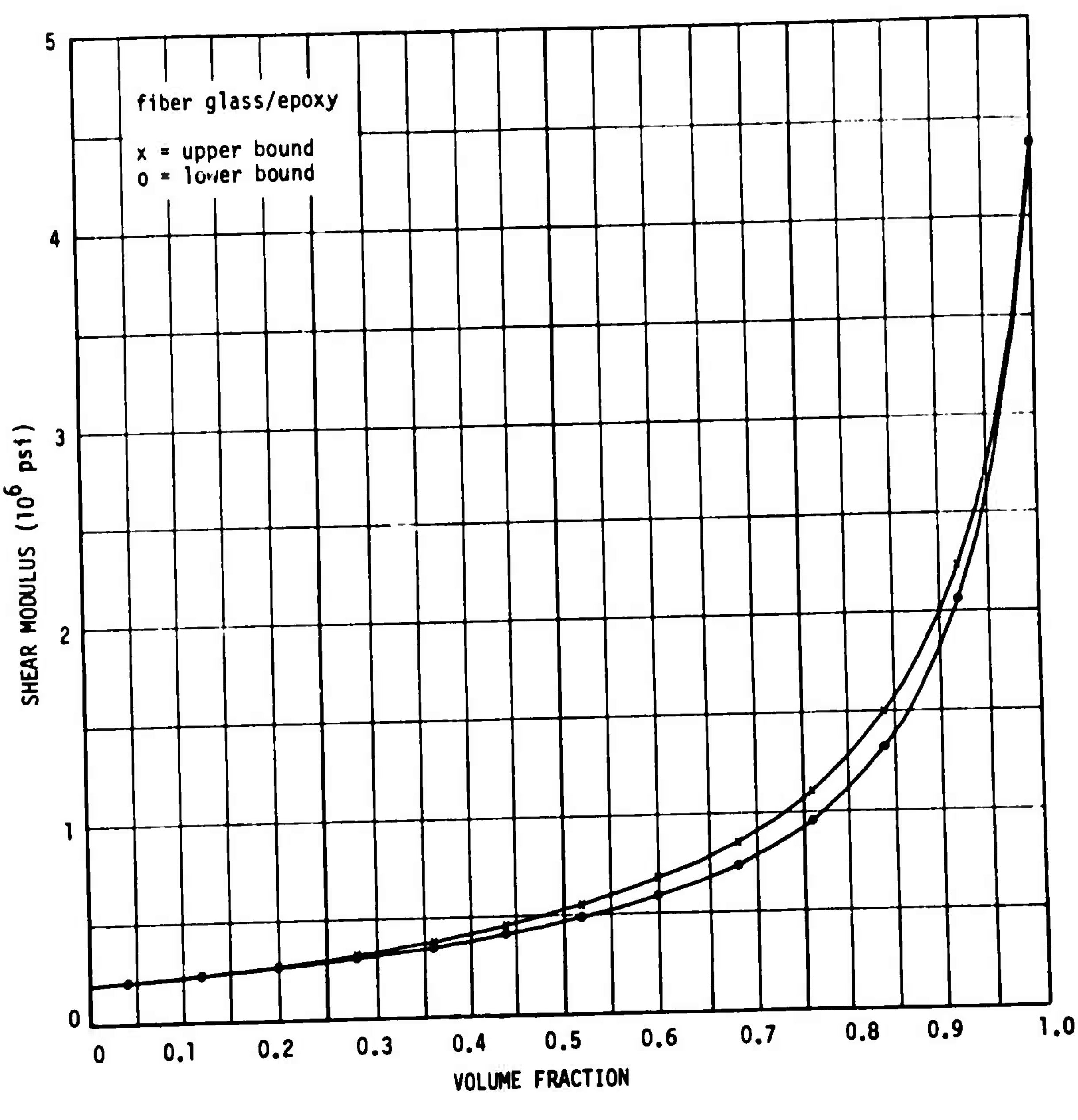

Figure 5. Shear Modulus G^*_{23} vs. Volume Fraction

ϕ = 30 degrees

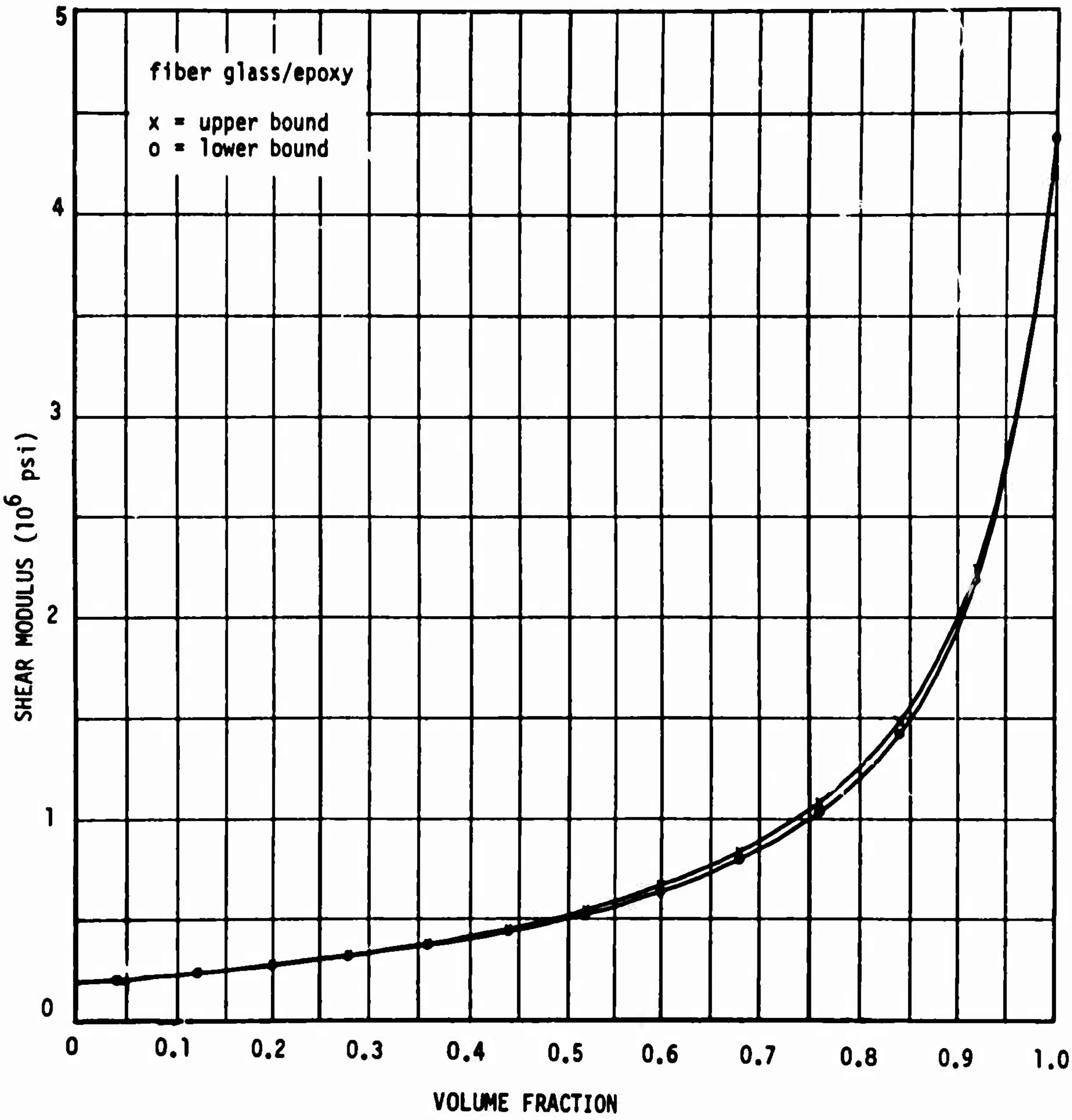

Figure 6. Shear Modulus G_{12}^* vs. Volume Fraction
ϕ = 30 degrees

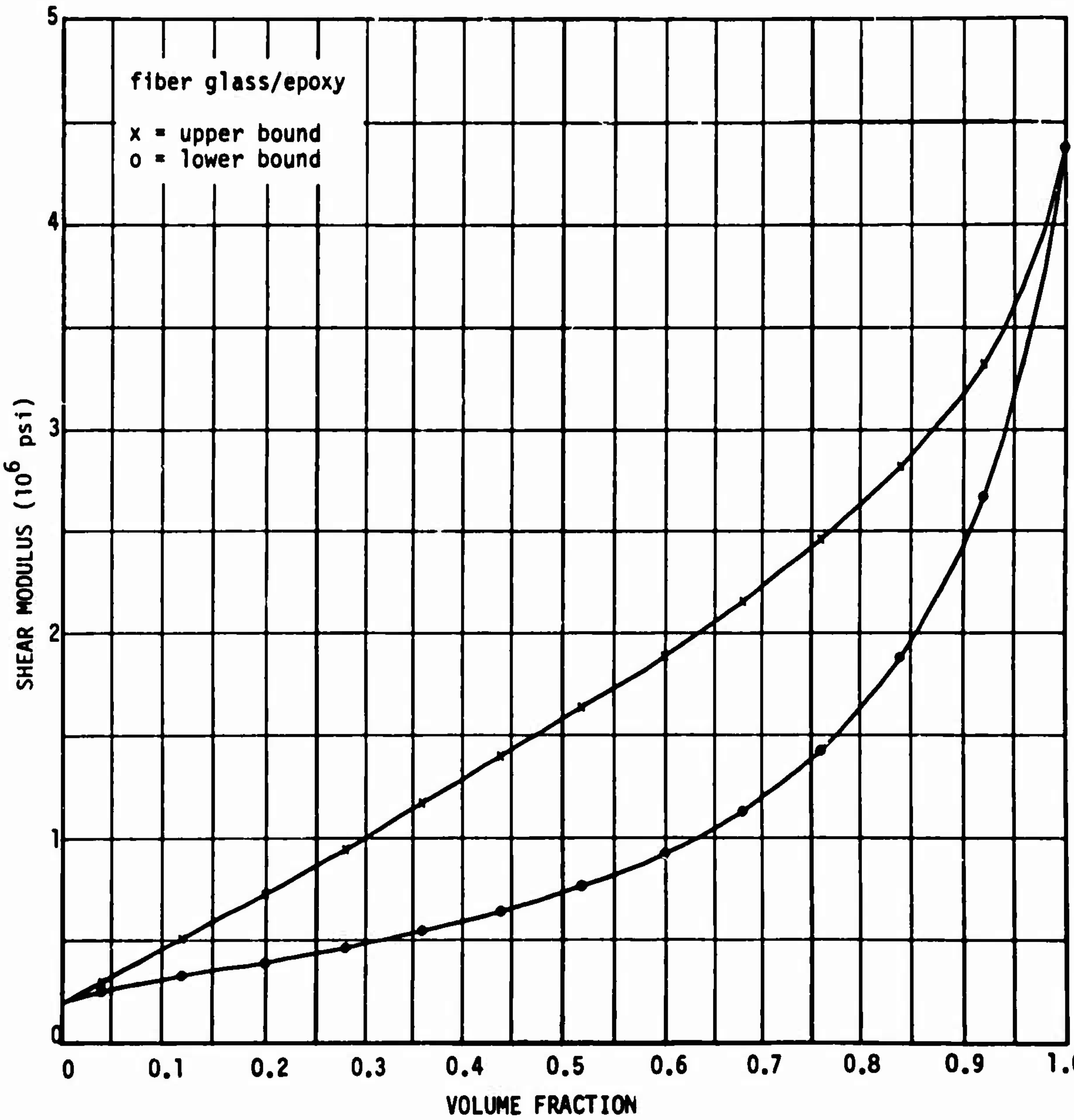

Figure 7. Shear Modulus G^*_{31} vs. Volume Fraction
ϕ = 45 degrees

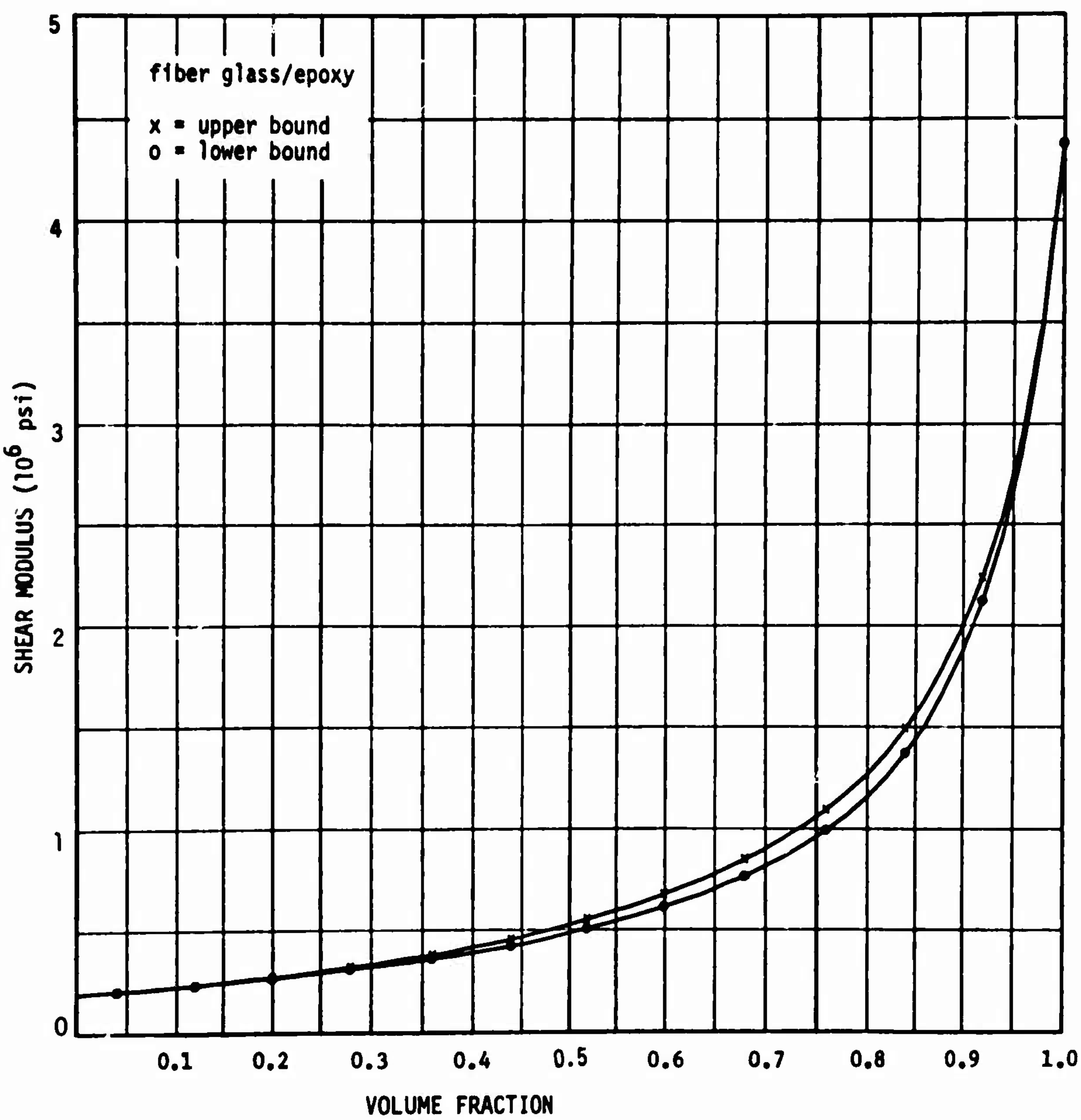

Figure 8. Shear Modulus G^*_{23} vs. Volume Fraction
ϕ = 45 degrees

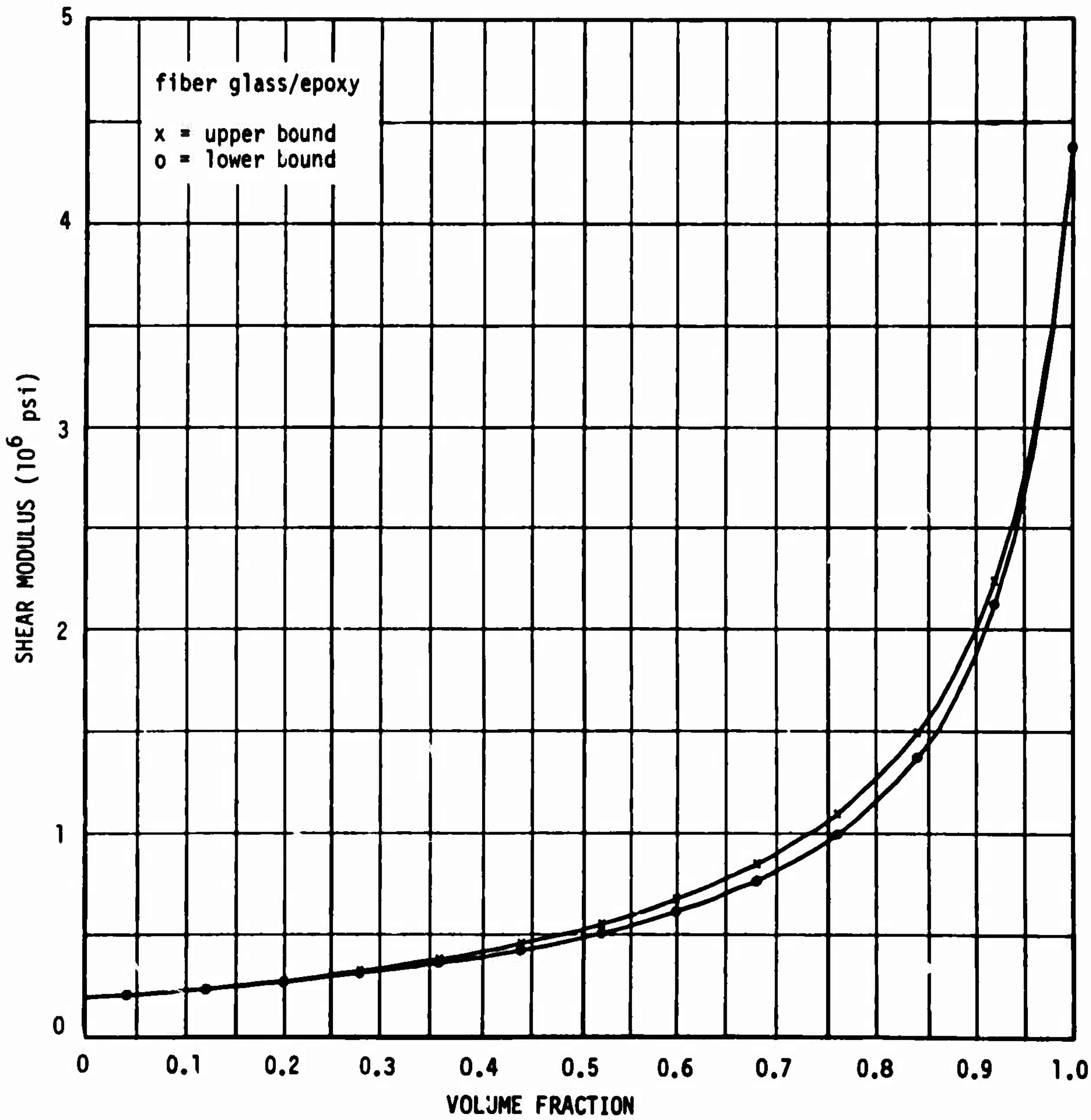

Figure 9. Shear Modulus G^*_{12} vs. Volume Fraction
ϕ = 45 degrees

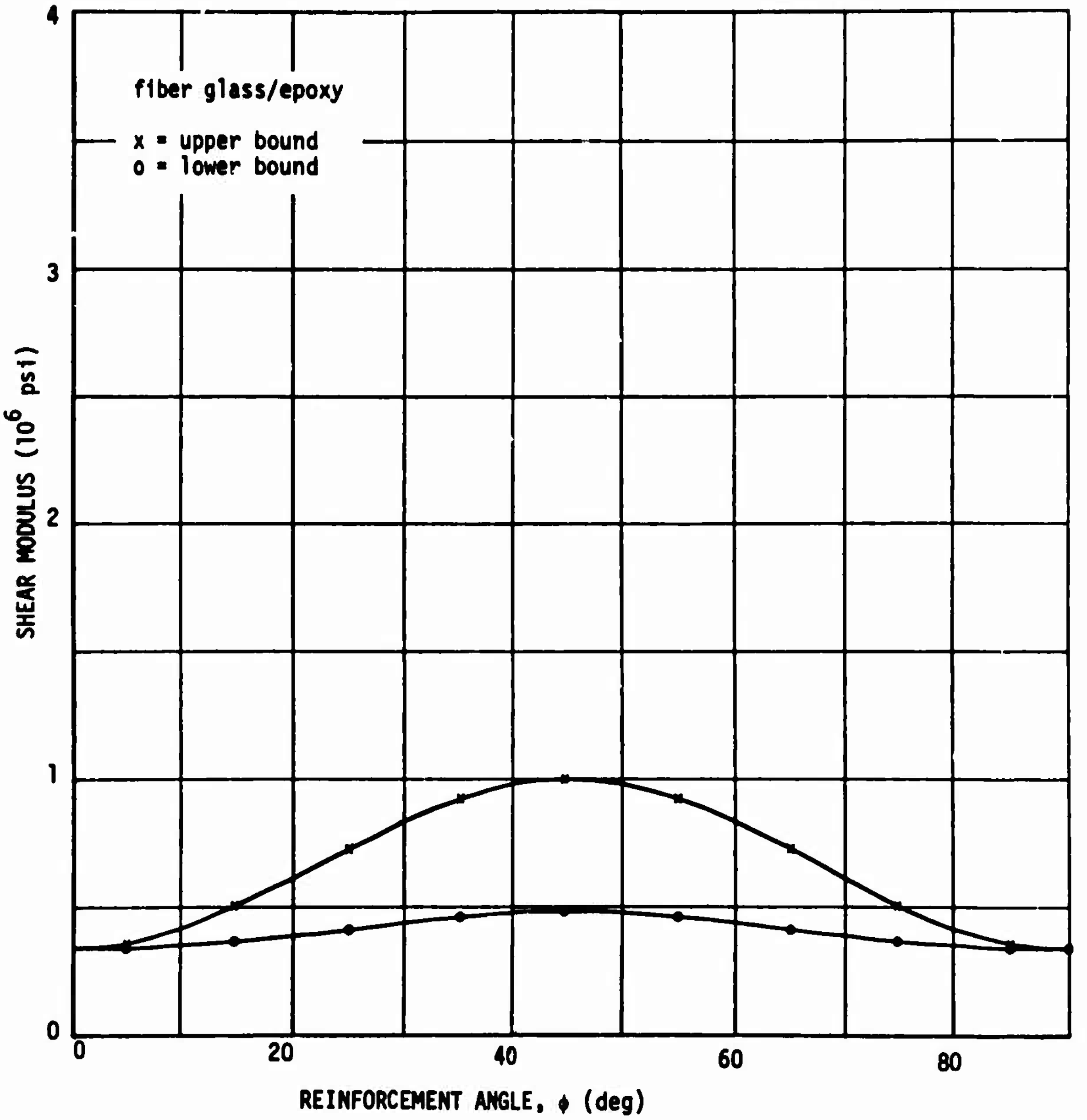

Figure 10. Shear Modulus G^*_{31} vs. Reinforcement Angle
Volume Fraction = 0.3

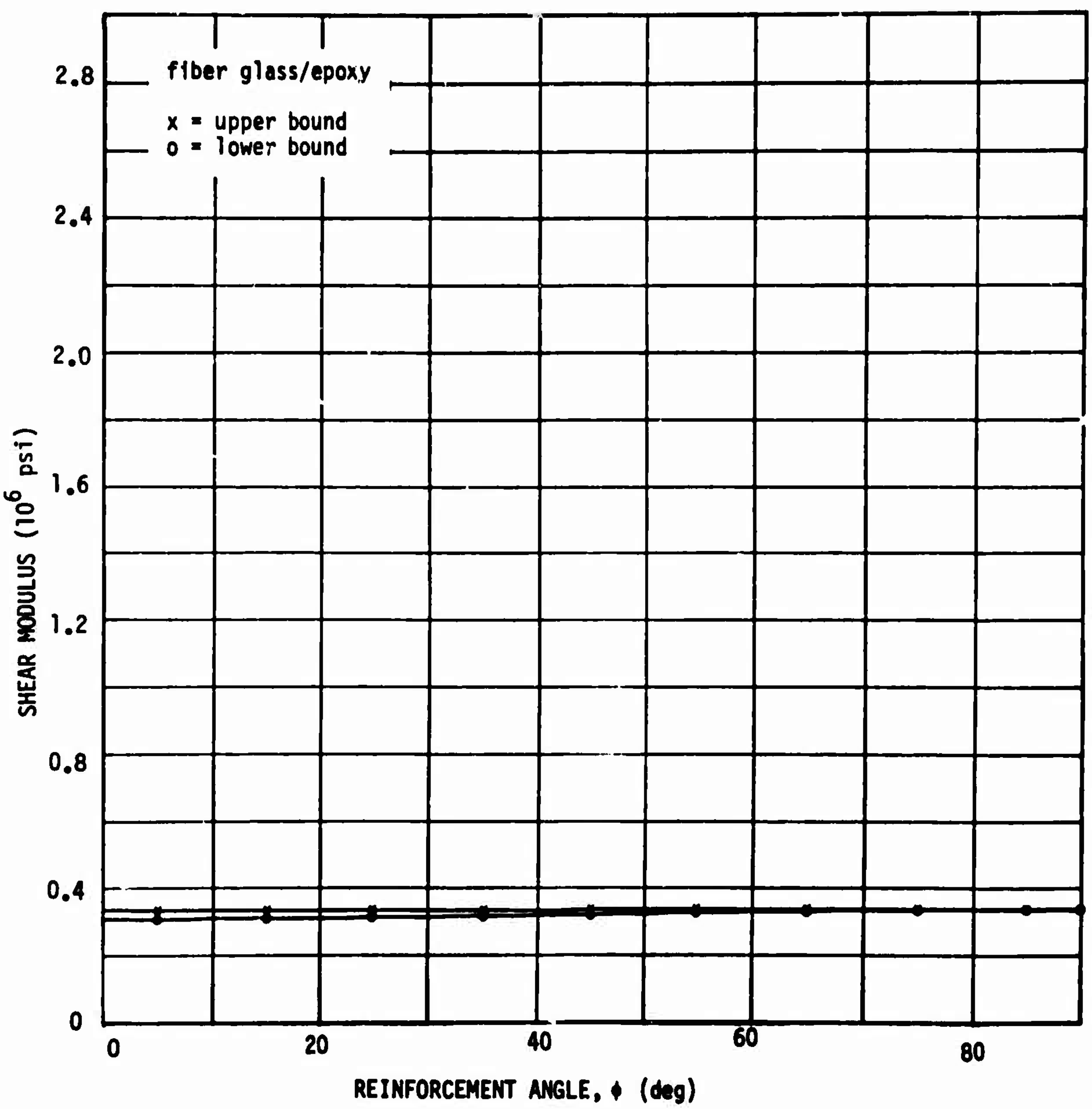

Figure 11. Shear Modulus G_{23}^{*} vs. Reinforcement Angle
Volume Fraction = 0.3

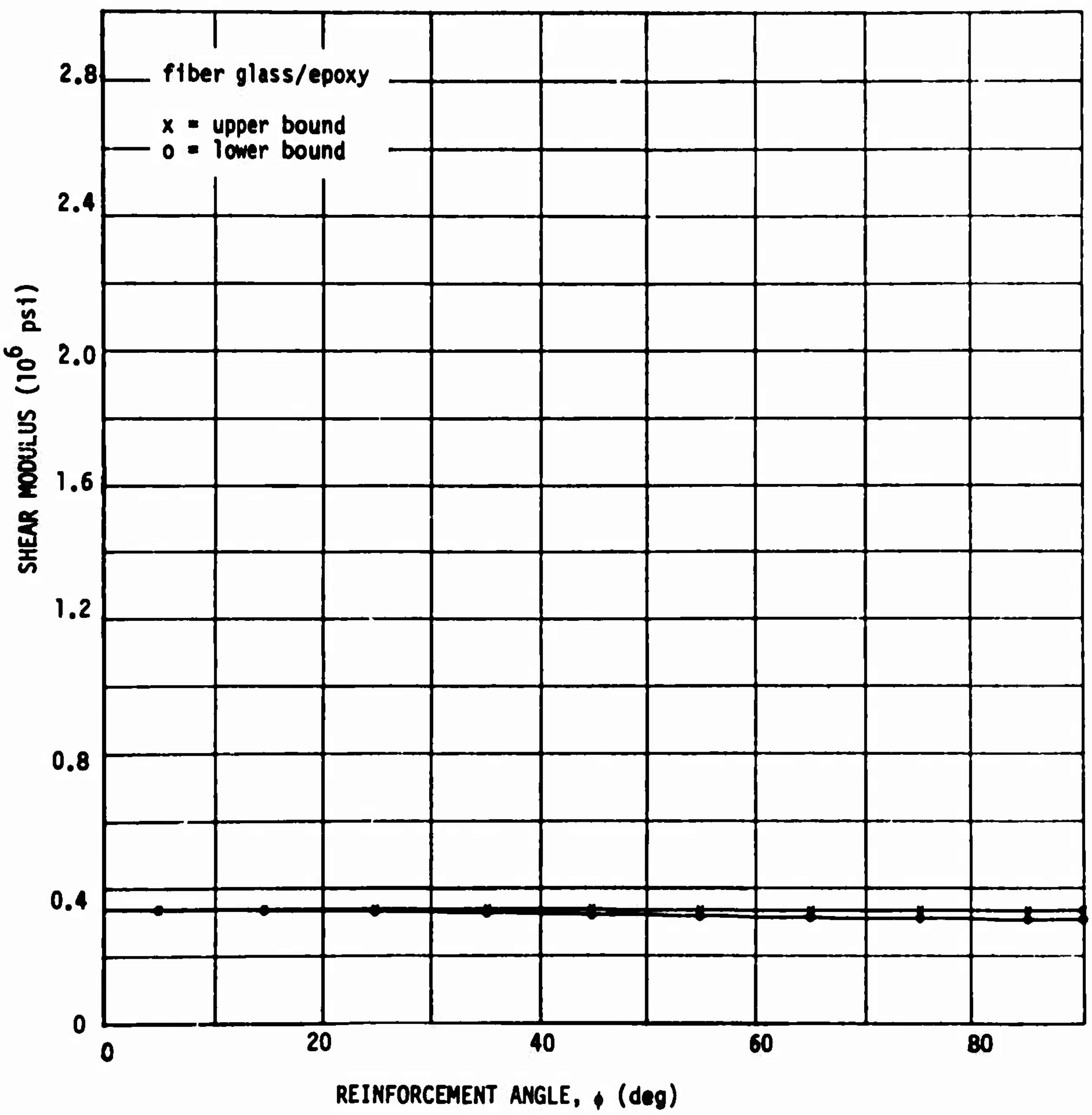

Figure 12. Shear Modulus G_{12}^* vs. Reinforcement Angle
Volume Fraction = 0.3

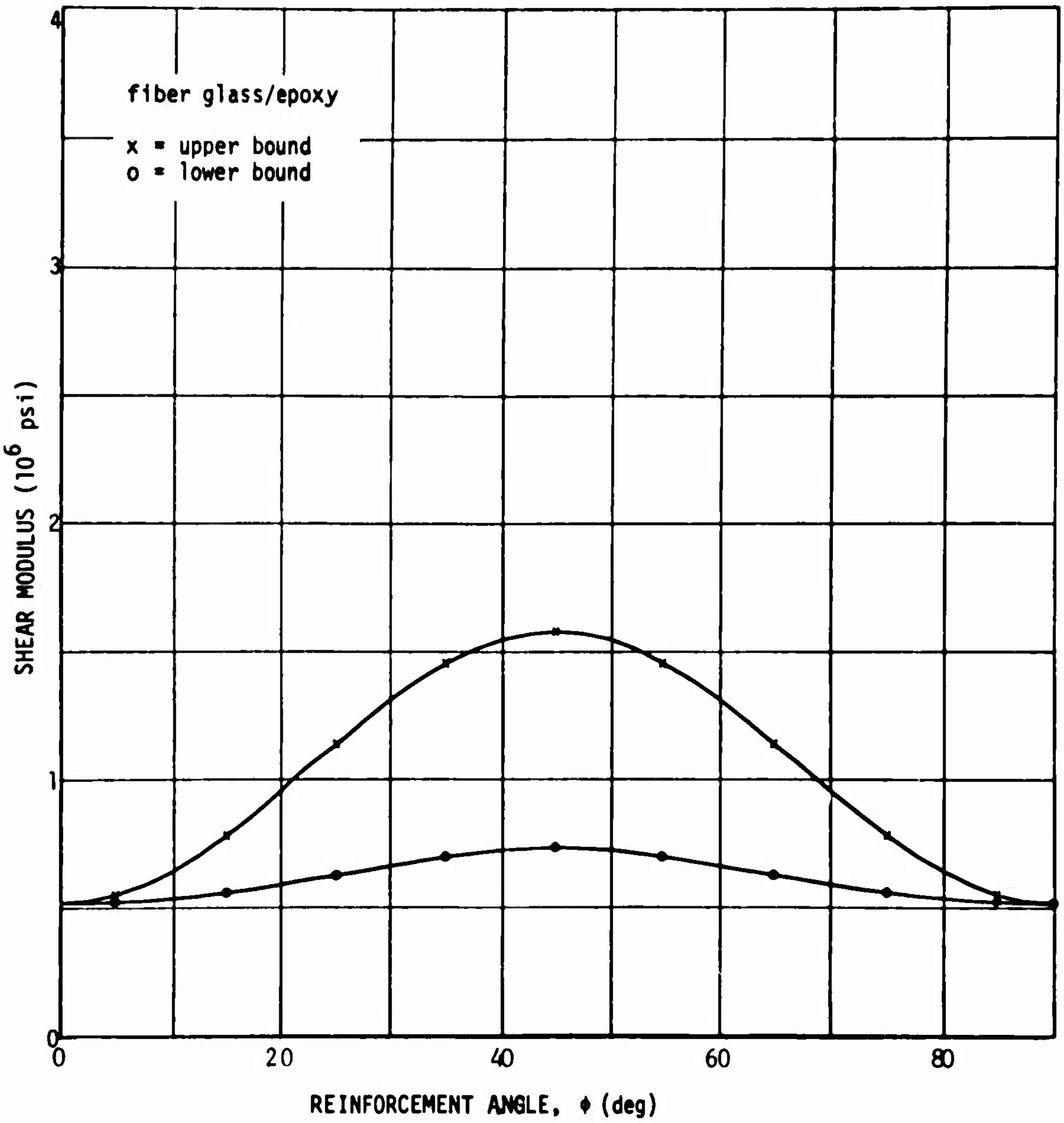

Figure 13. Shear Modulus G^*_{31} vs. Reinforcement Angle
Volume Fraction = 0.5

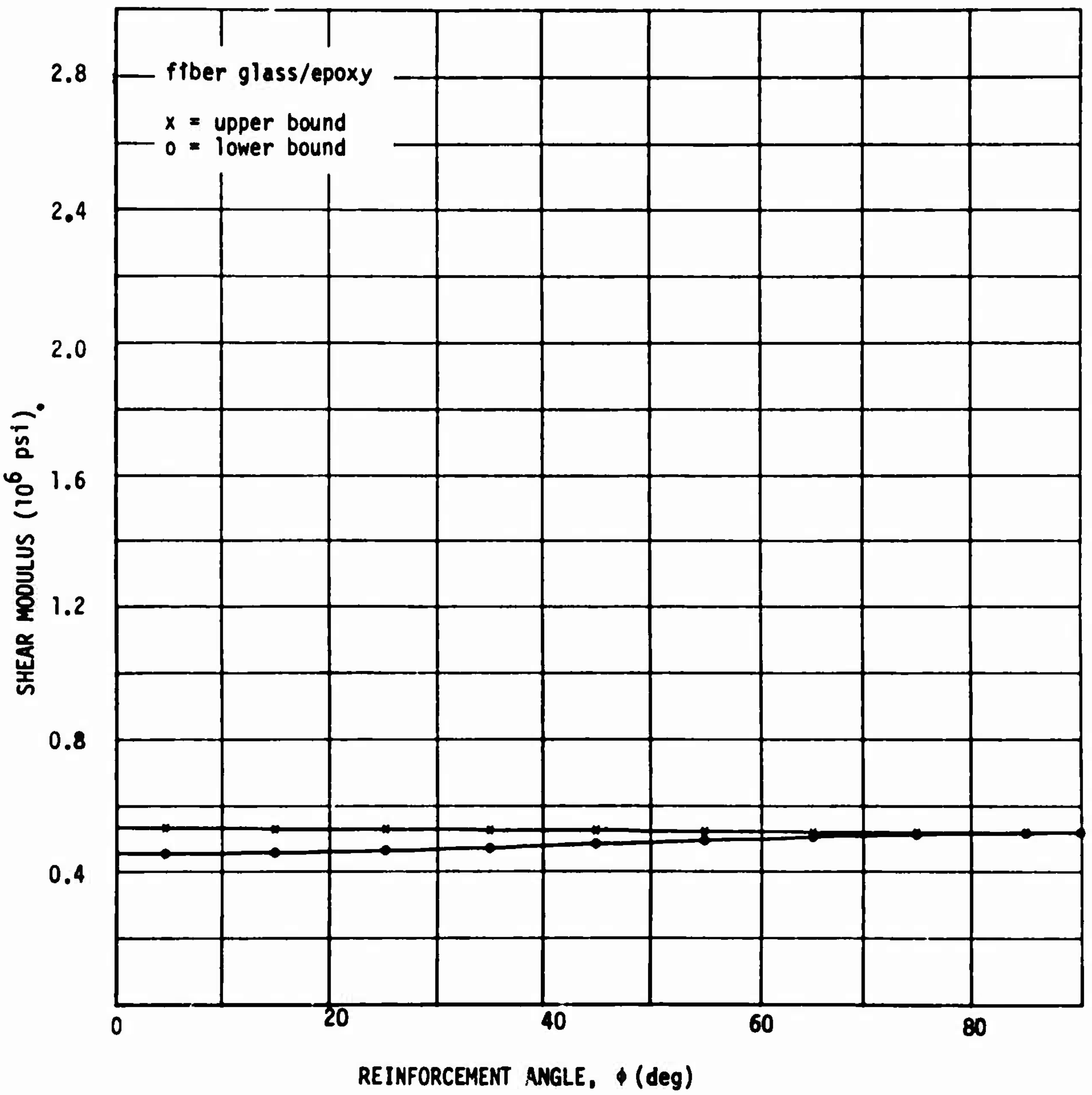

Figure 14. Shear Modulus G_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.5

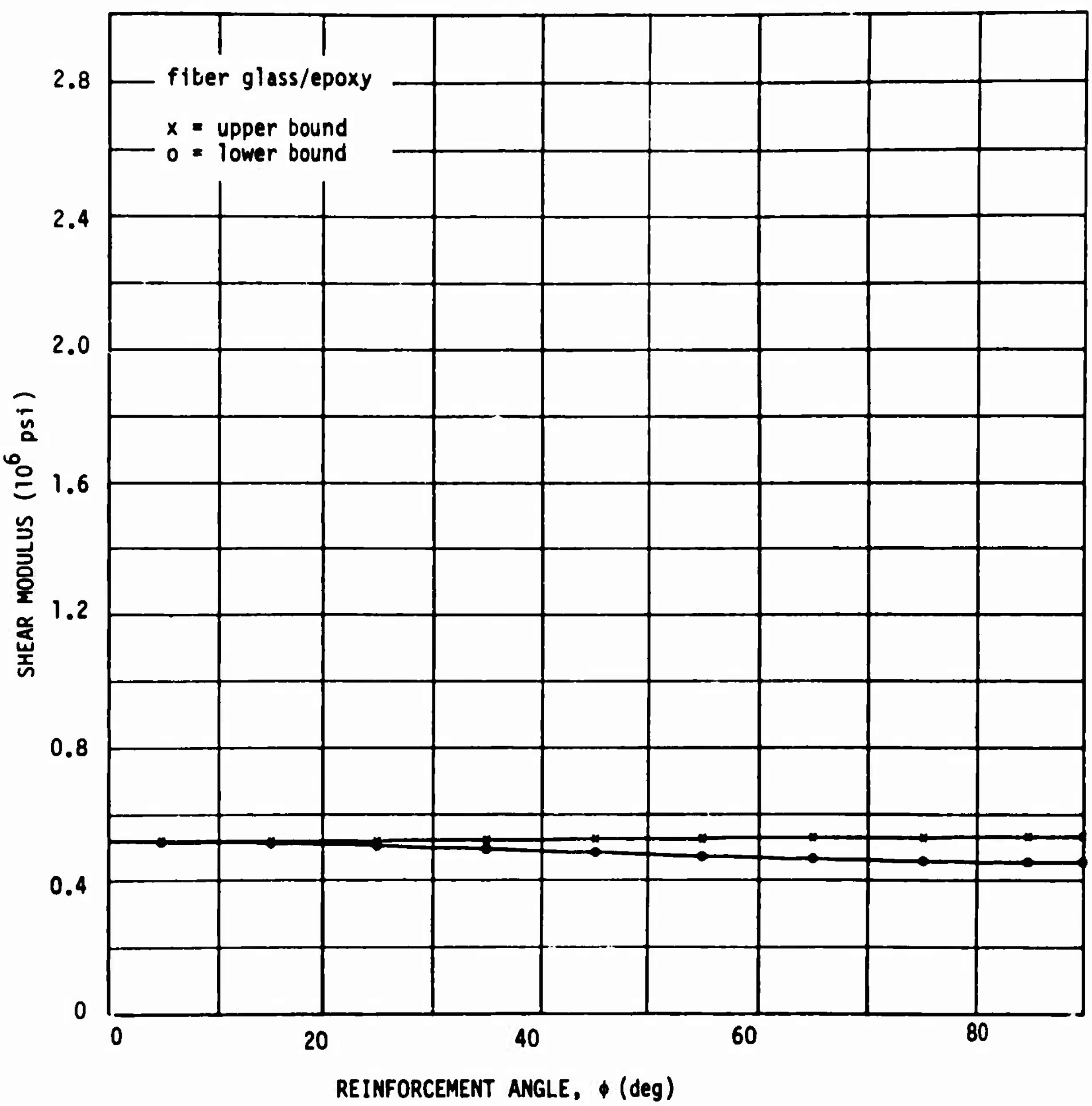

Figure 15. Shear Modulus G^*_{12} vs. Reinforcement Angle
Volume Fraction = 0.5

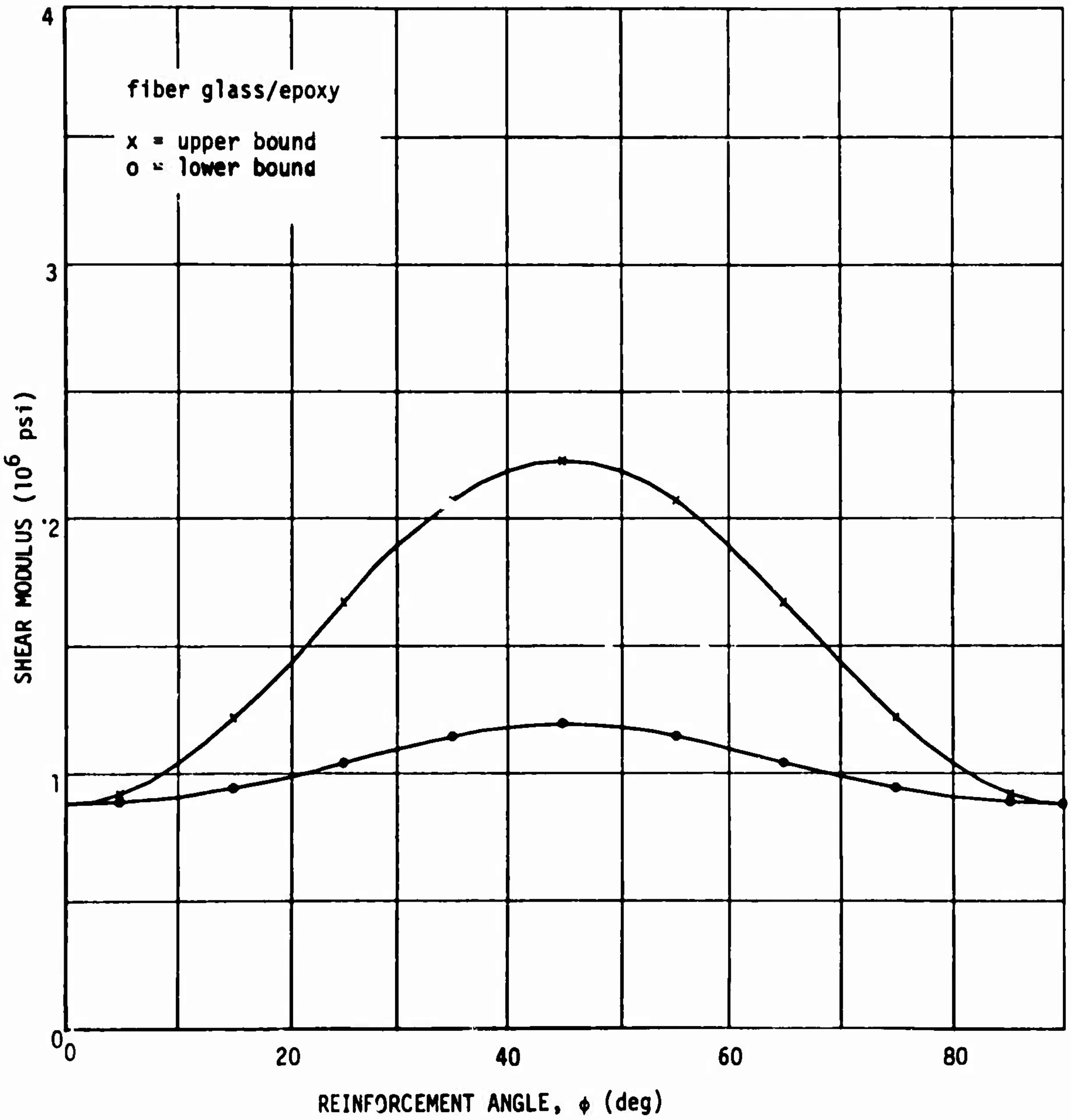

Figure 16. Shear Modulus G_{31}^* vs. Reinforcement Angle
Volume Fraction = 0.7

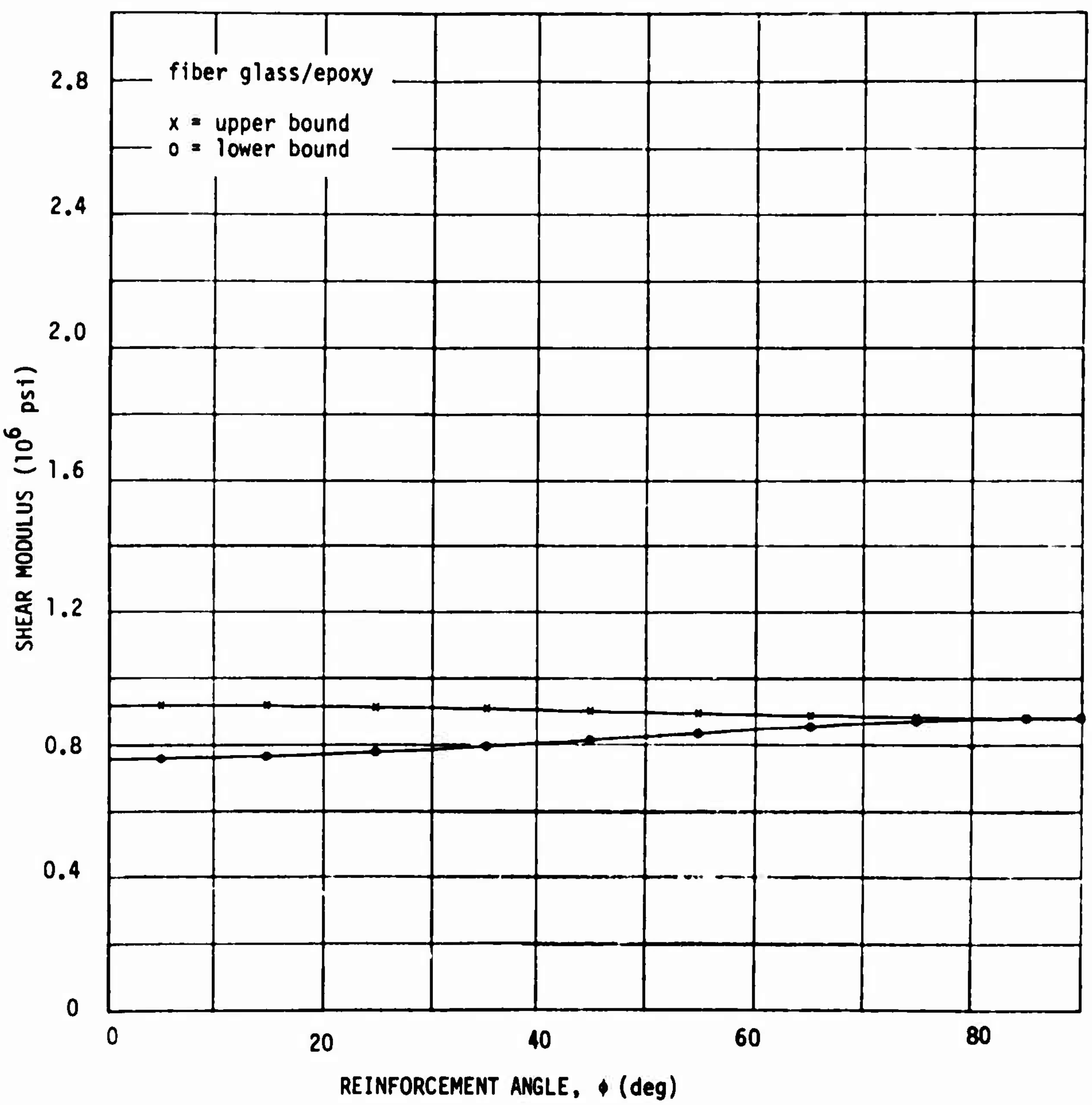

Figure 17. Shear Modulus G_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.7

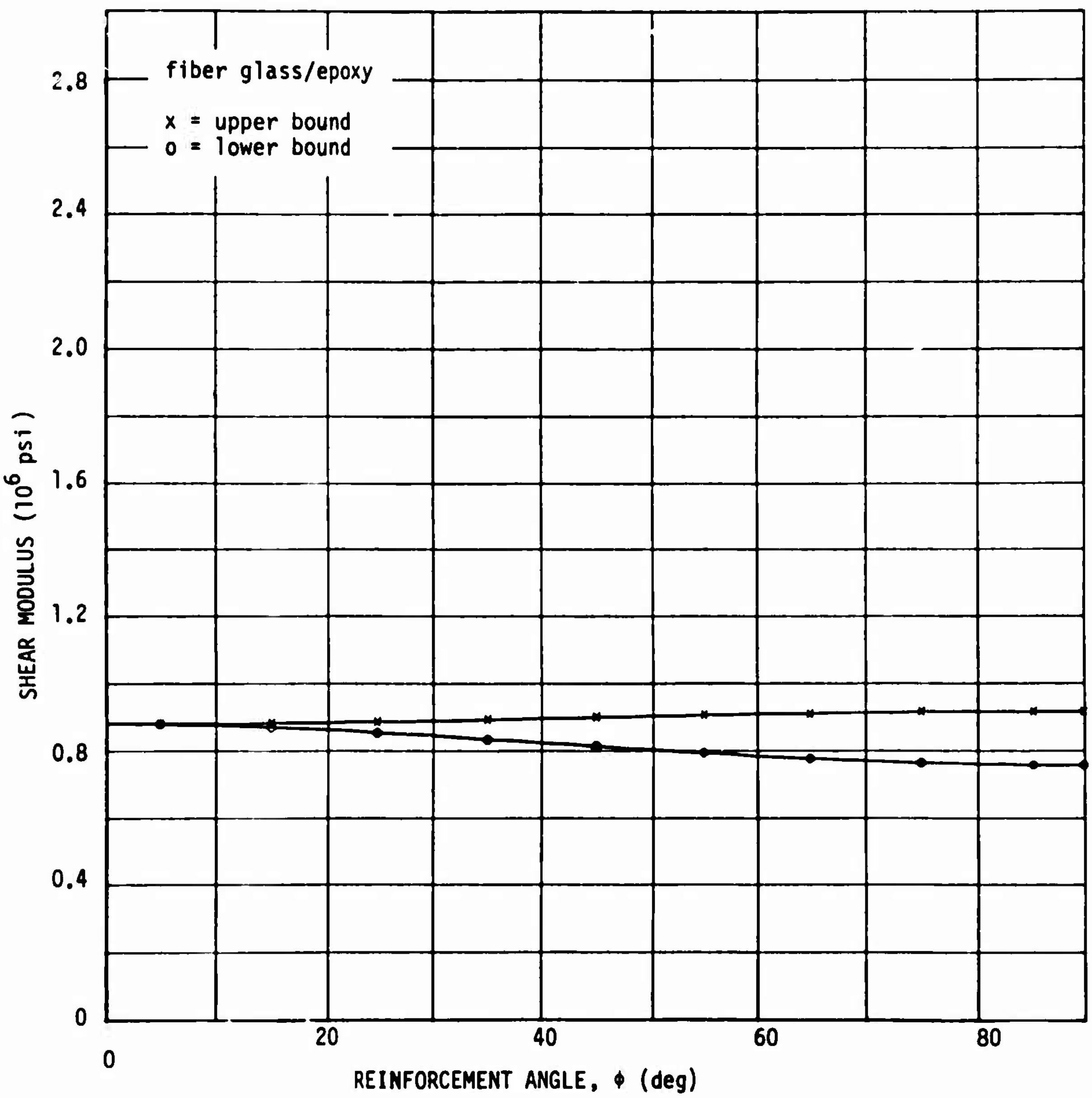

Figure 18. Shear Modulus G^*_{12} vs. Reinforcement Angle
Volume Fraction = 0.7

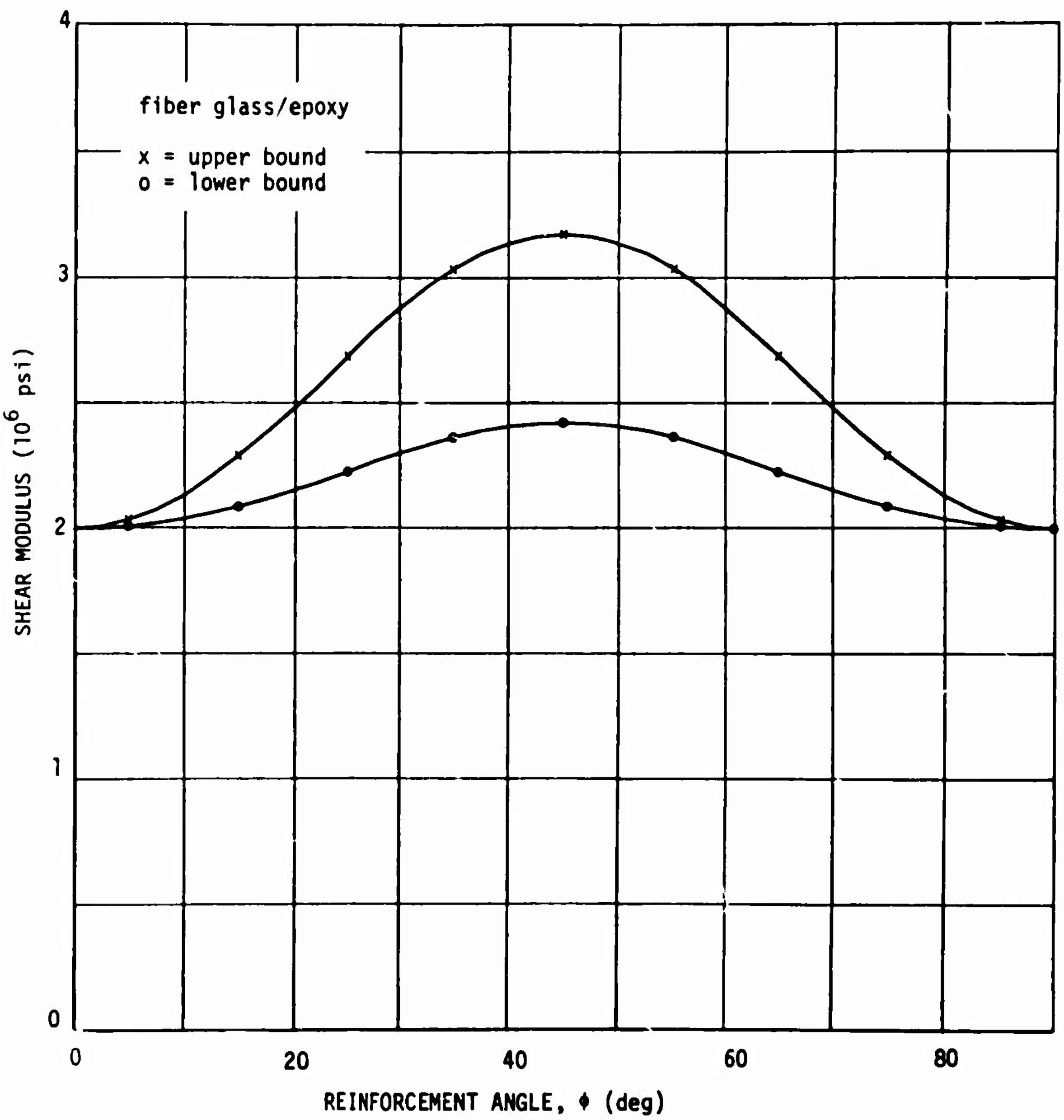

Figure 19. Shear Modulus G^*_{31} vs. Reinforcement Angle
Volume Fraction = 0.9

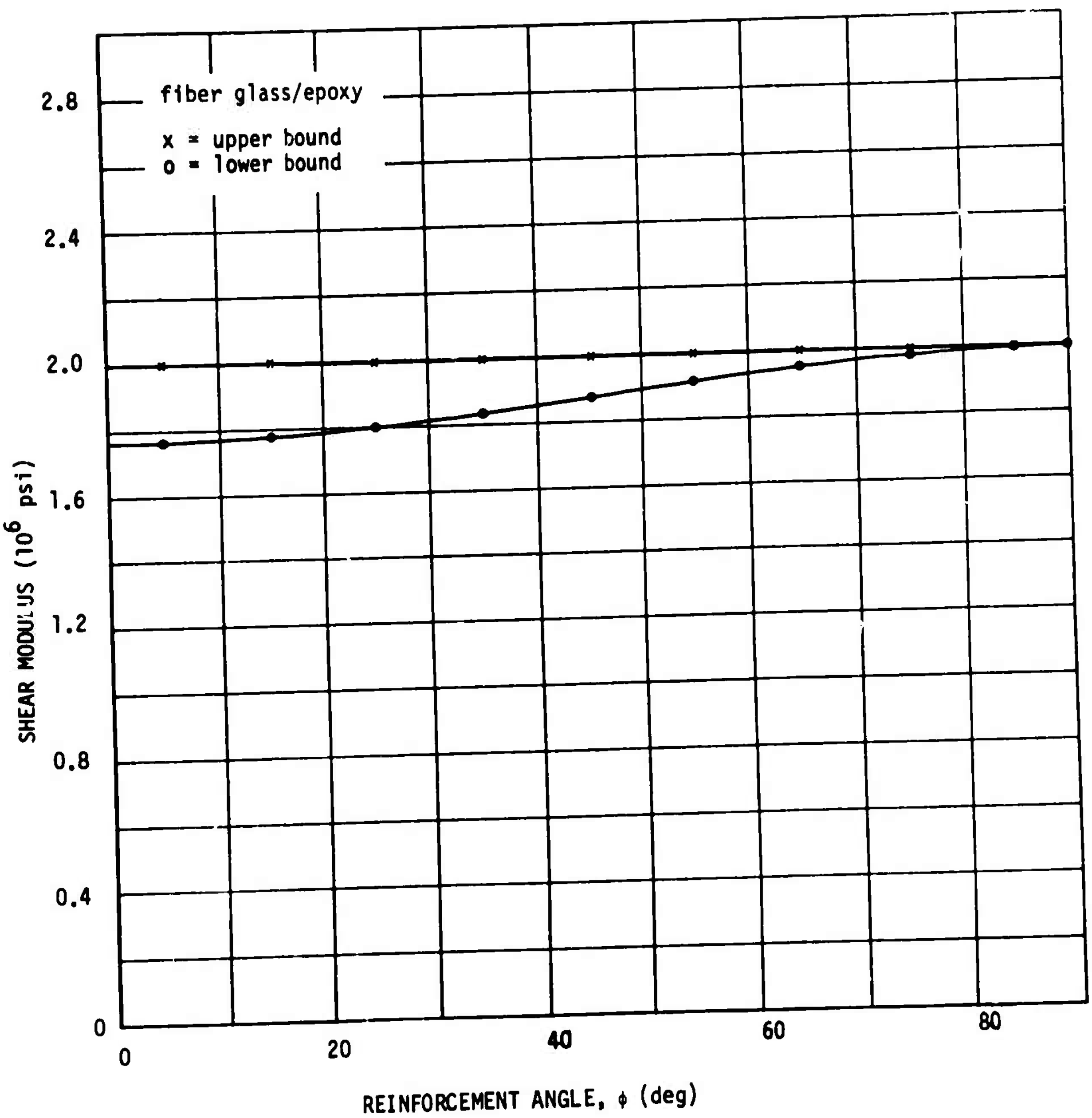

Figure 20. Shear Modulus G_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.9

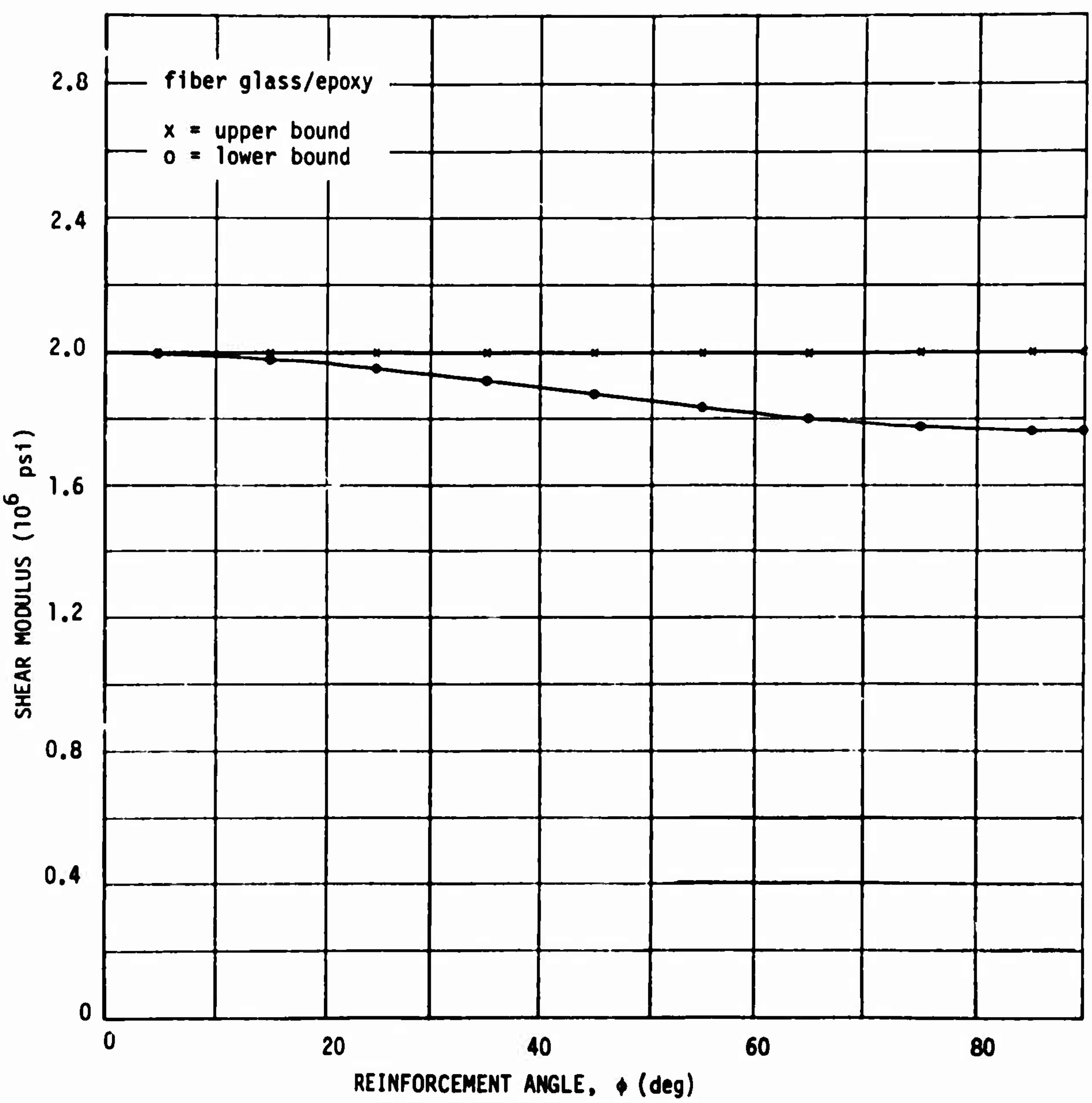

Figure 21. Shear Modulus G_{12}^* vs. Reinforcement Angle
Volume Fraction = 0.9

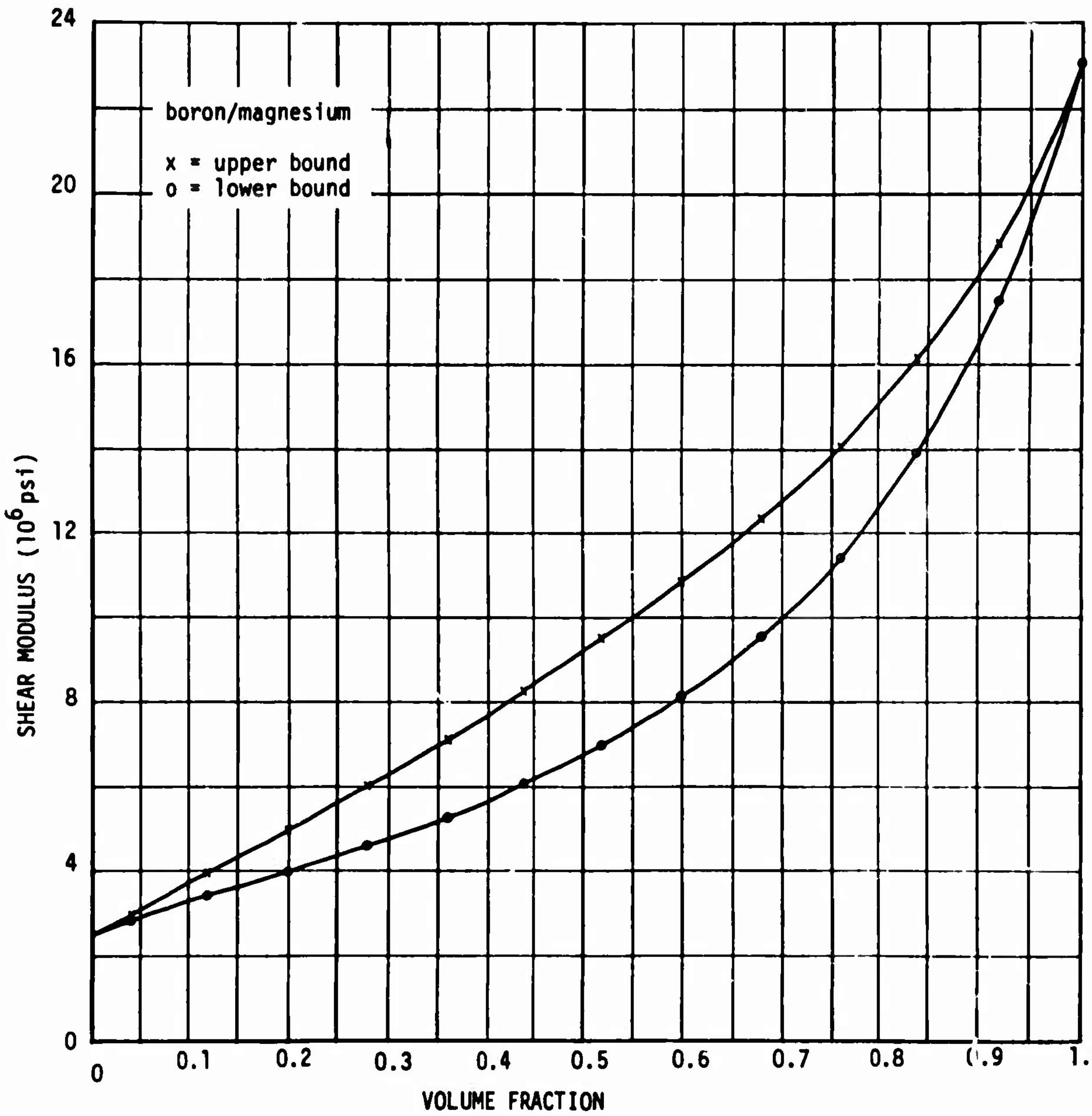

Figure 22. Shear Modulus G^*_{31} vs. Volume Fraction
ϕ = 30 degrees

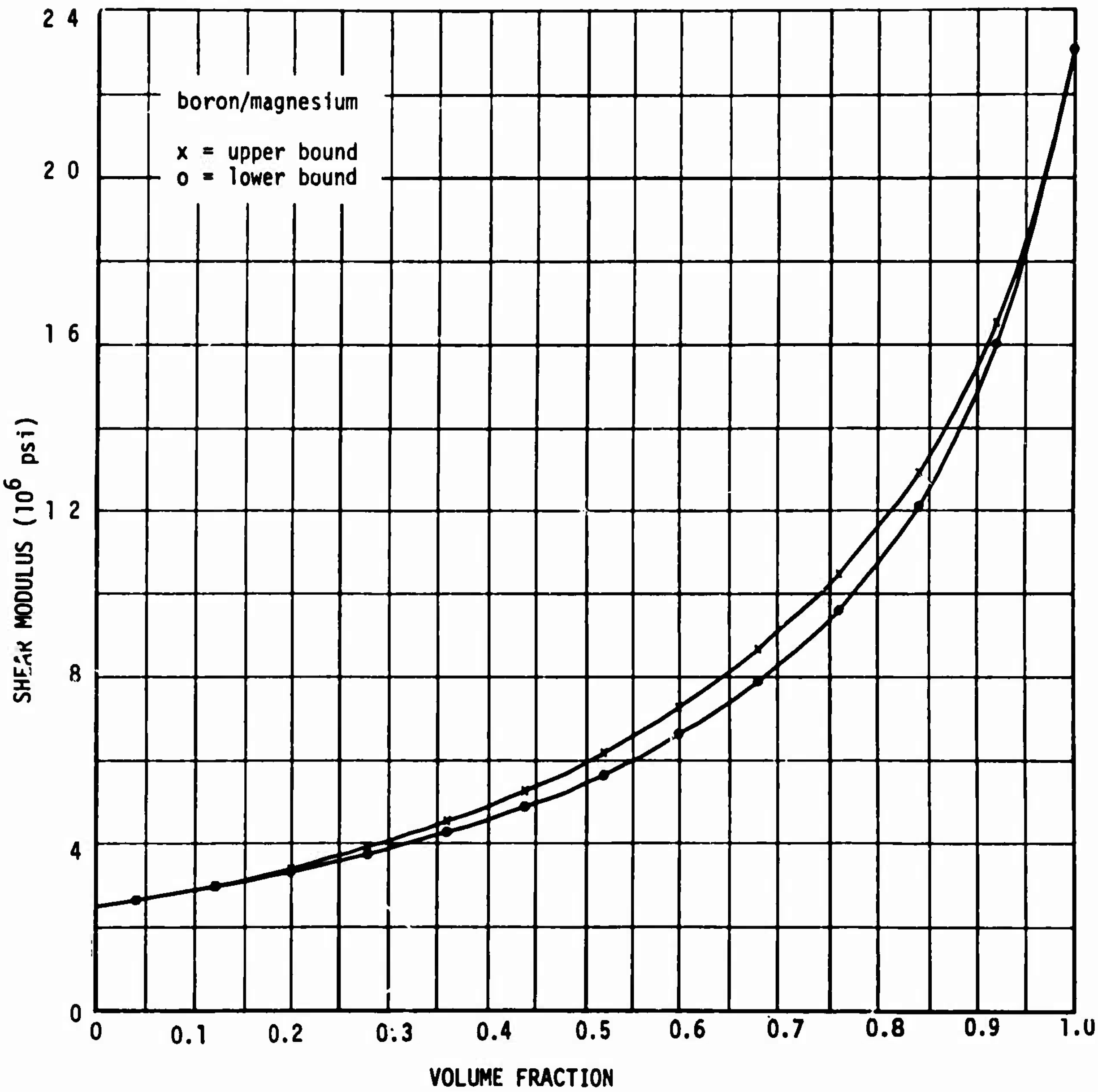

Figure 23. Shear Modulus G_{23}^* vs. Volume Fraction
ϕ = 30 degrees

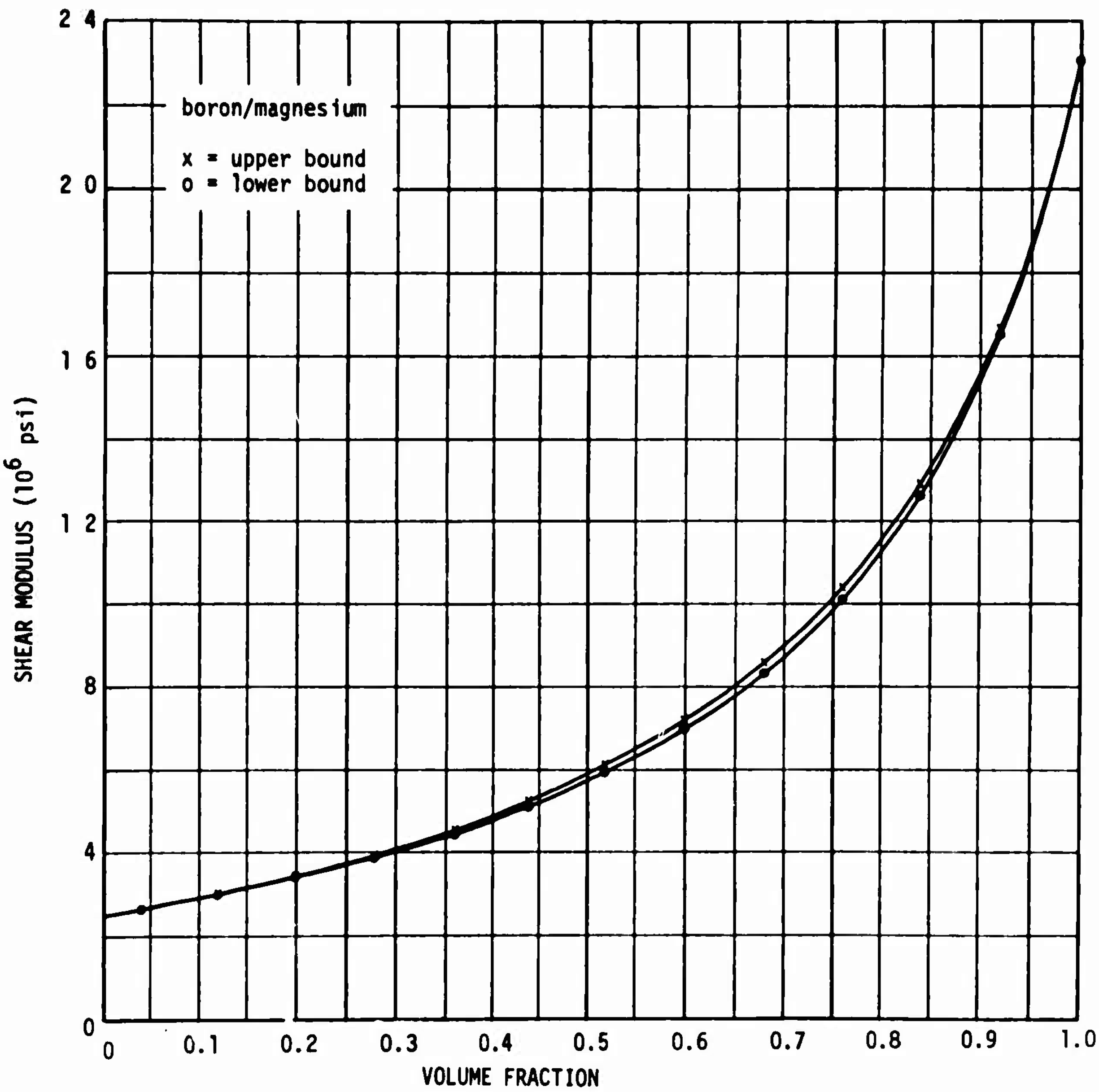

Figure 24. Shear Modulus G_{12}^* vs. Volume Fraction
ϕ = 30 degrees

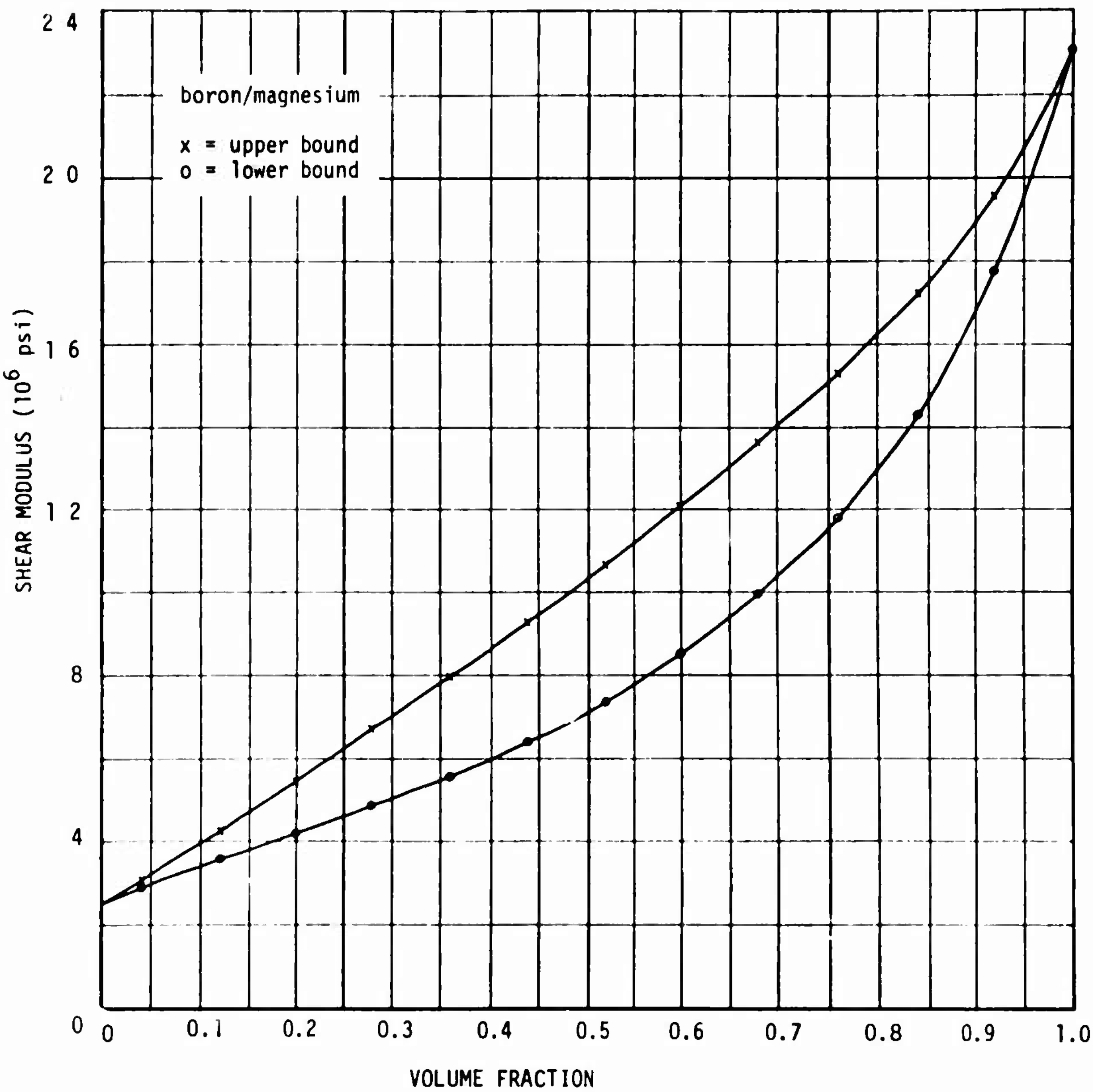

Figure 25. Shear Modulus G_{31}^* vs. Volume Fraction
ϕ = 45 degrees

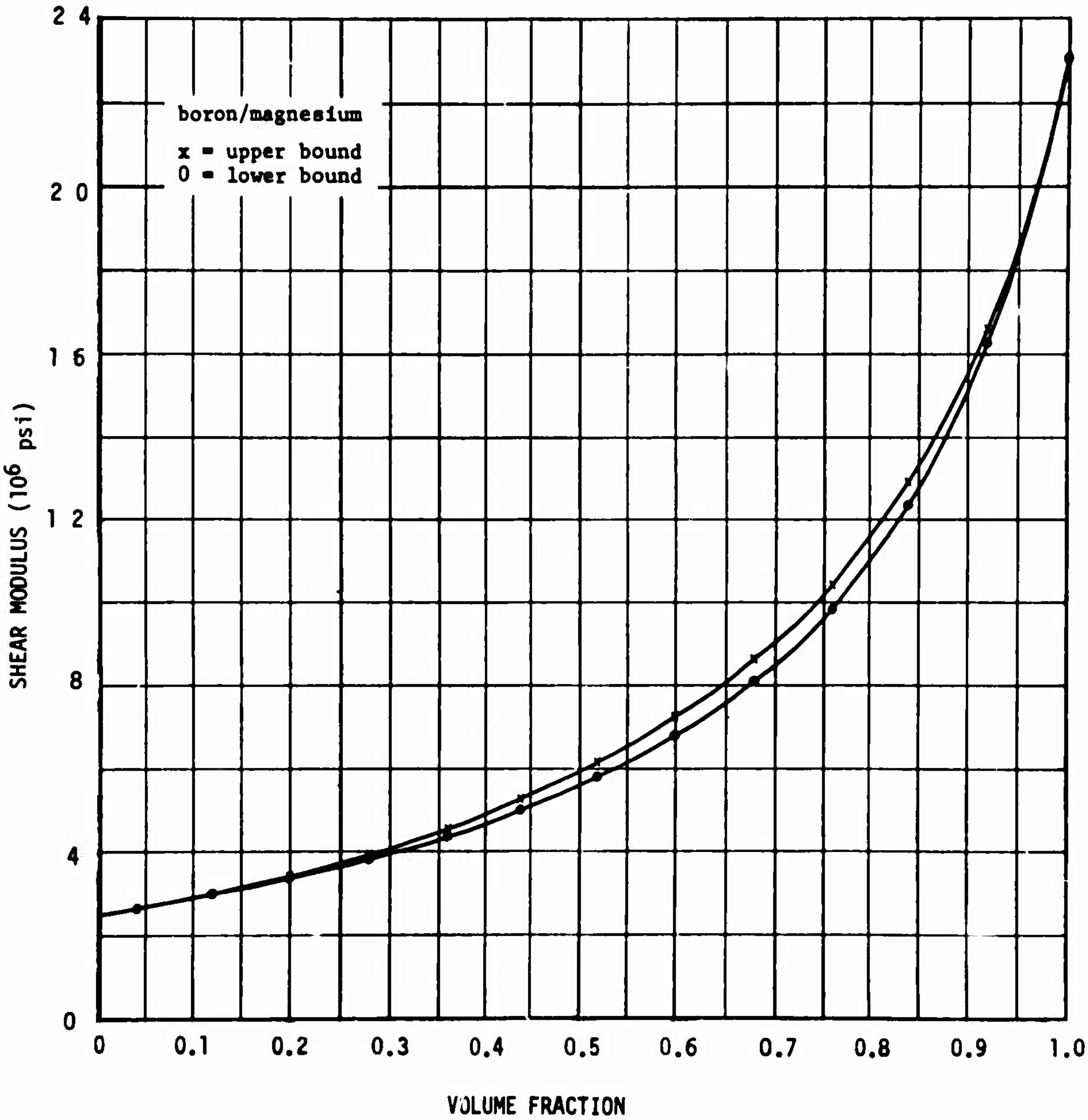

Figure 26. Shear Modulus G_{23}^* vs. Volume Fraction
ϕ = 45 degrees

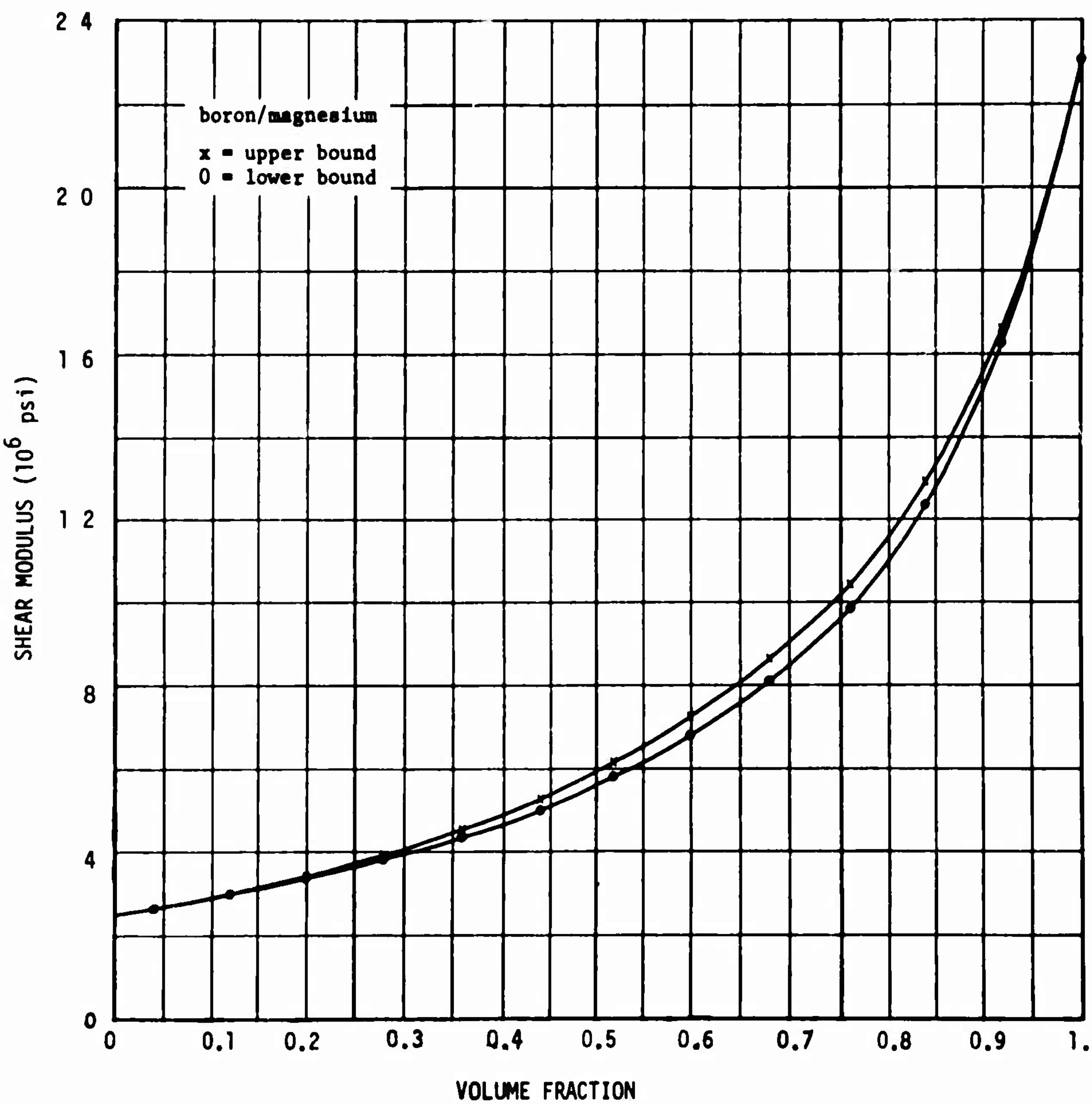

Figure 27. Shear Modulus G_{12}^* vs. Volume Fraction
ϕ = 45 degrees

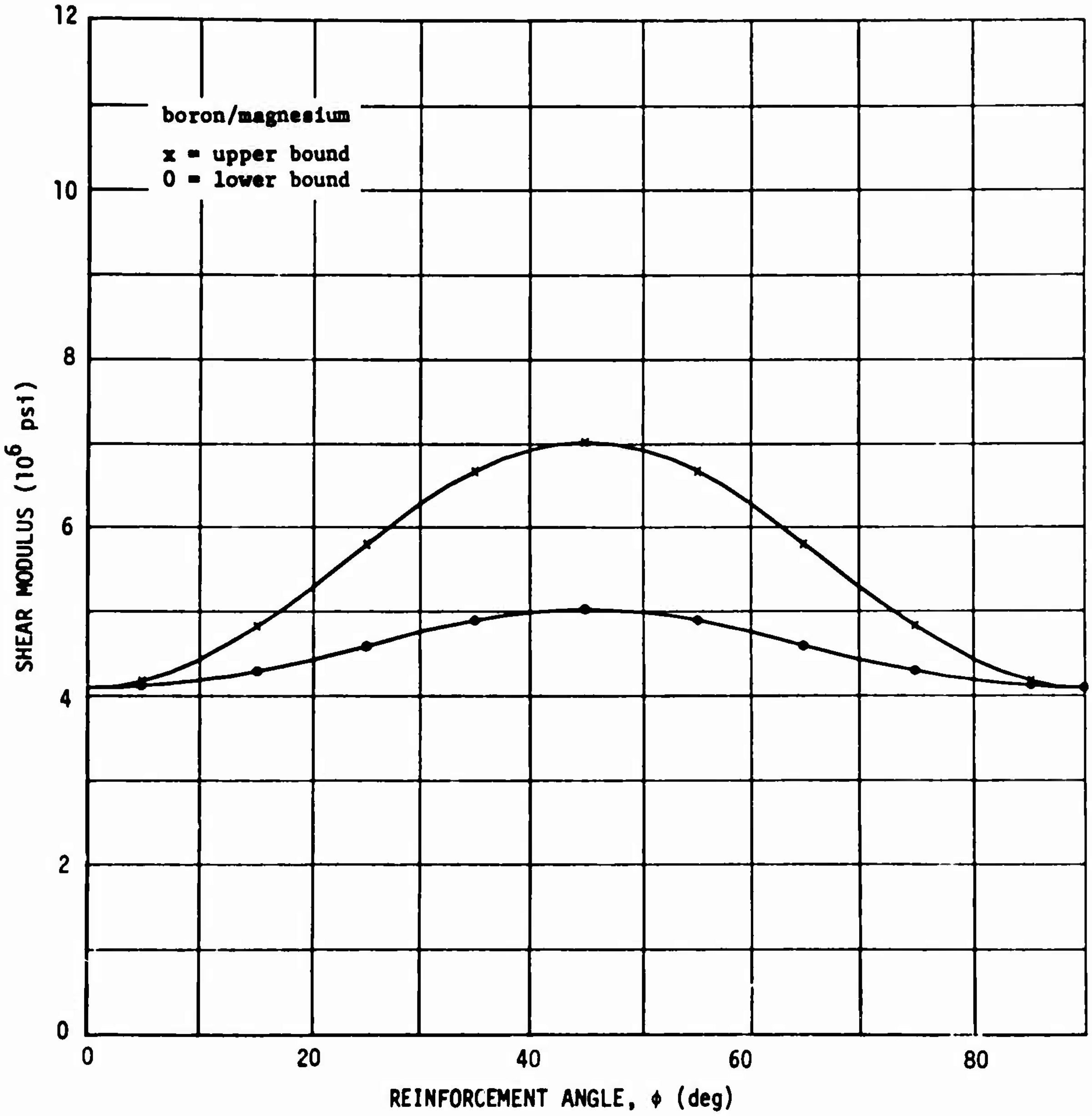

Figure 28. Shear Modulus G^*_{31} vs. Reinforcement Angle
Volume Fraction = 0.3

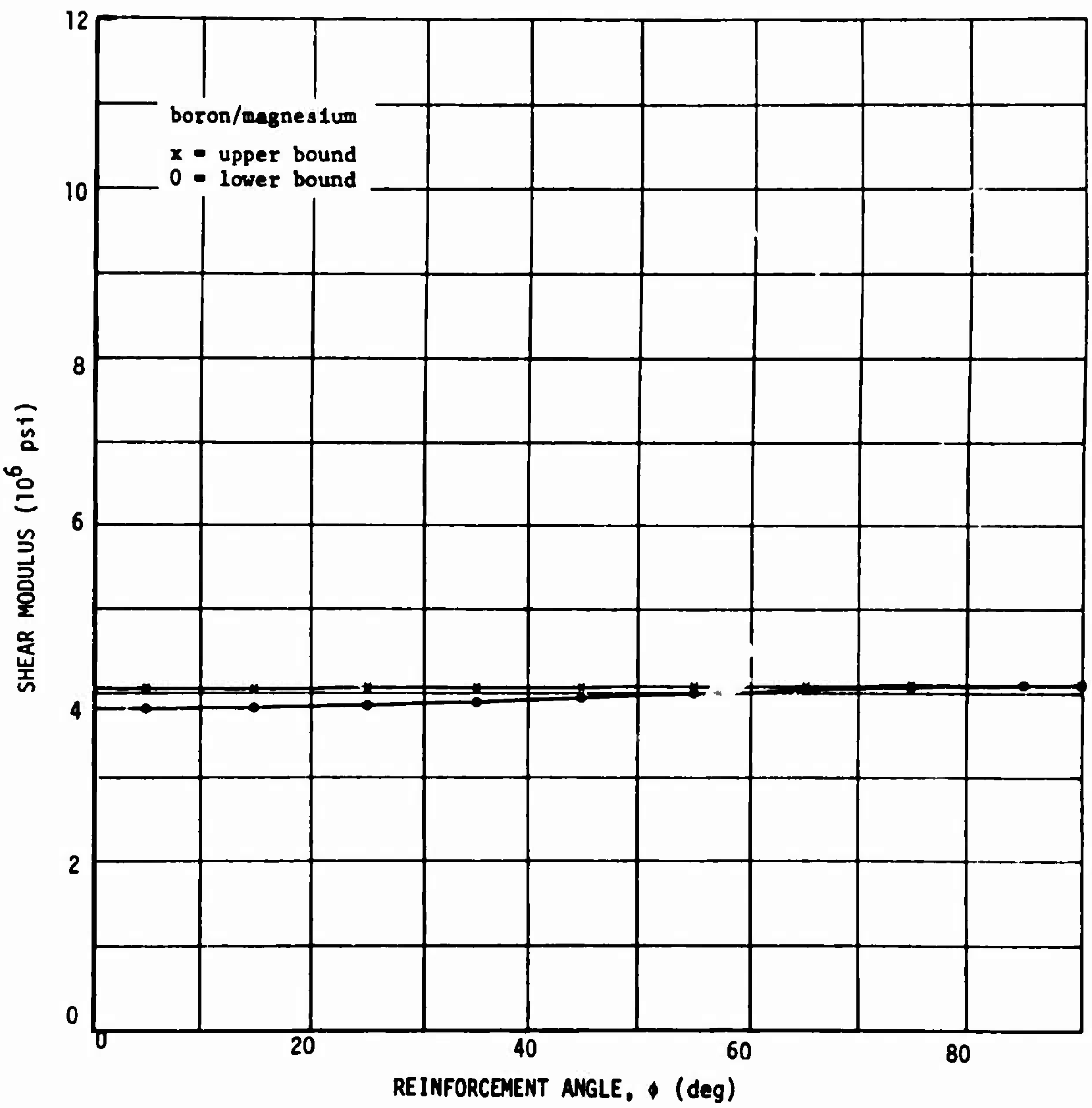

Figure 29. Shear Modulus G_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.3

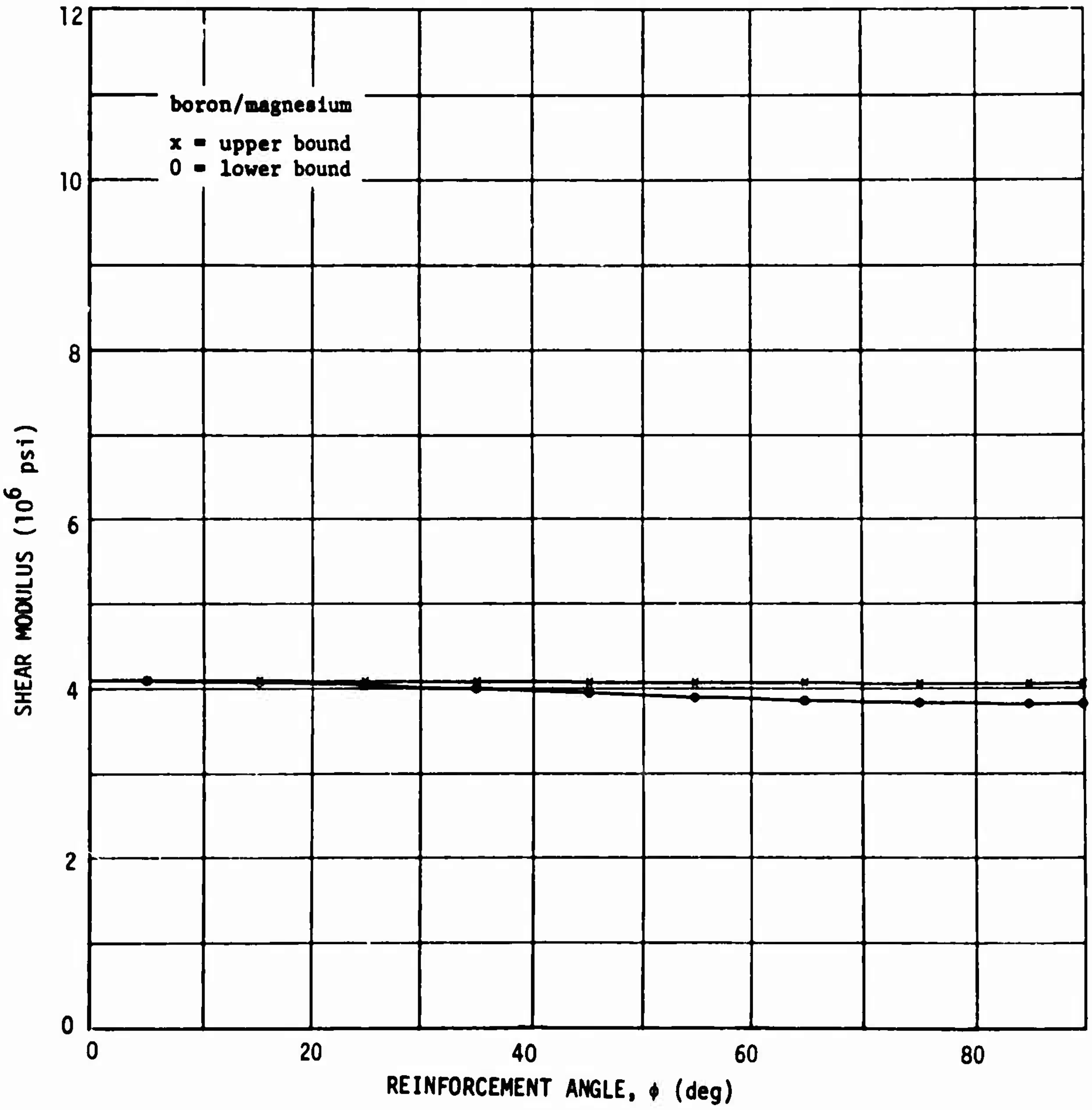

Figure 30. Shear Modulus G_{12}^* vs. Reinforcement Angle
Volume Fraction = 0.3

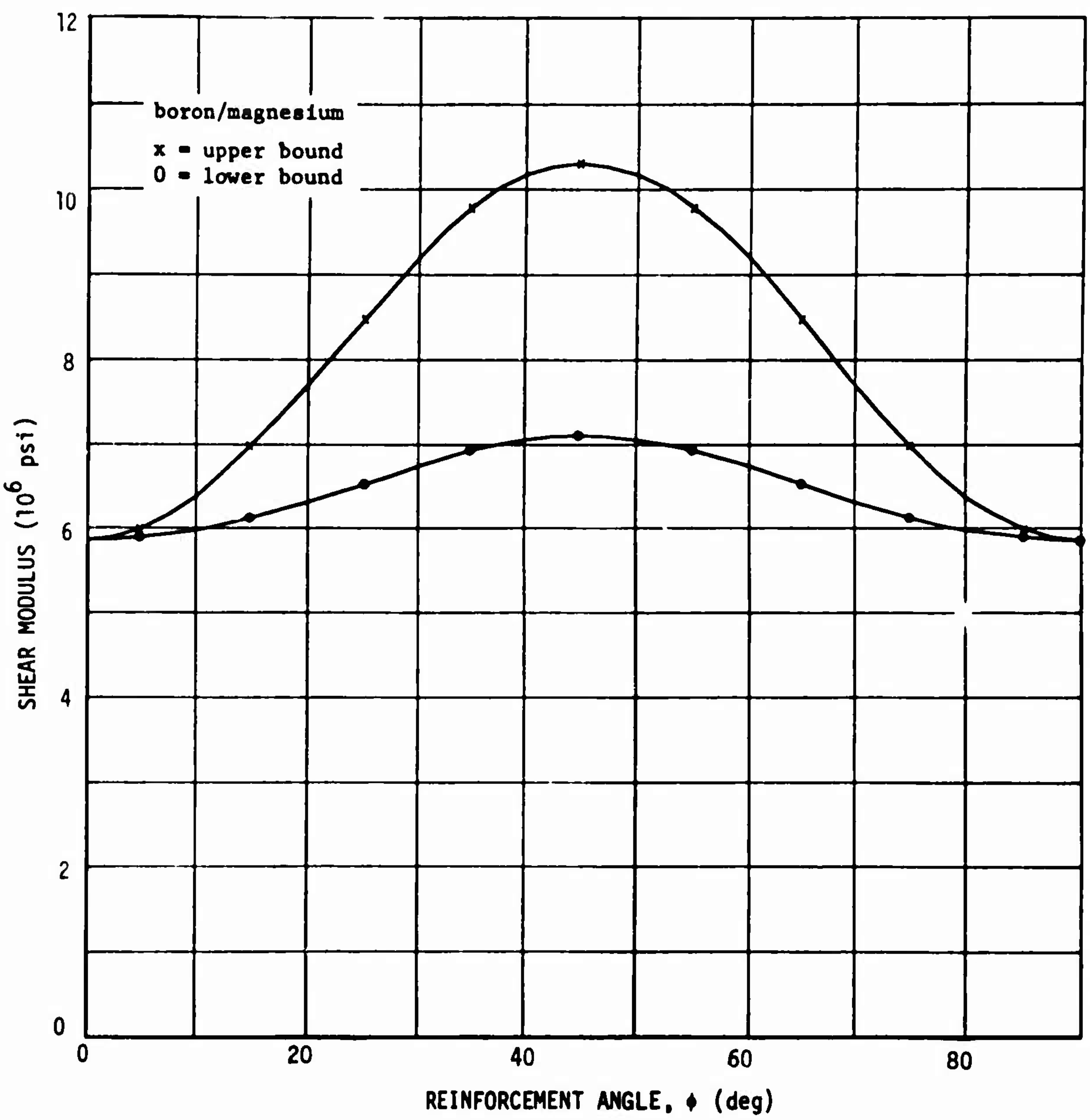

Figure 31. Shear Modulus G^*_{31} vs. Reinforcement Angle
Volume Fraction = 0.5

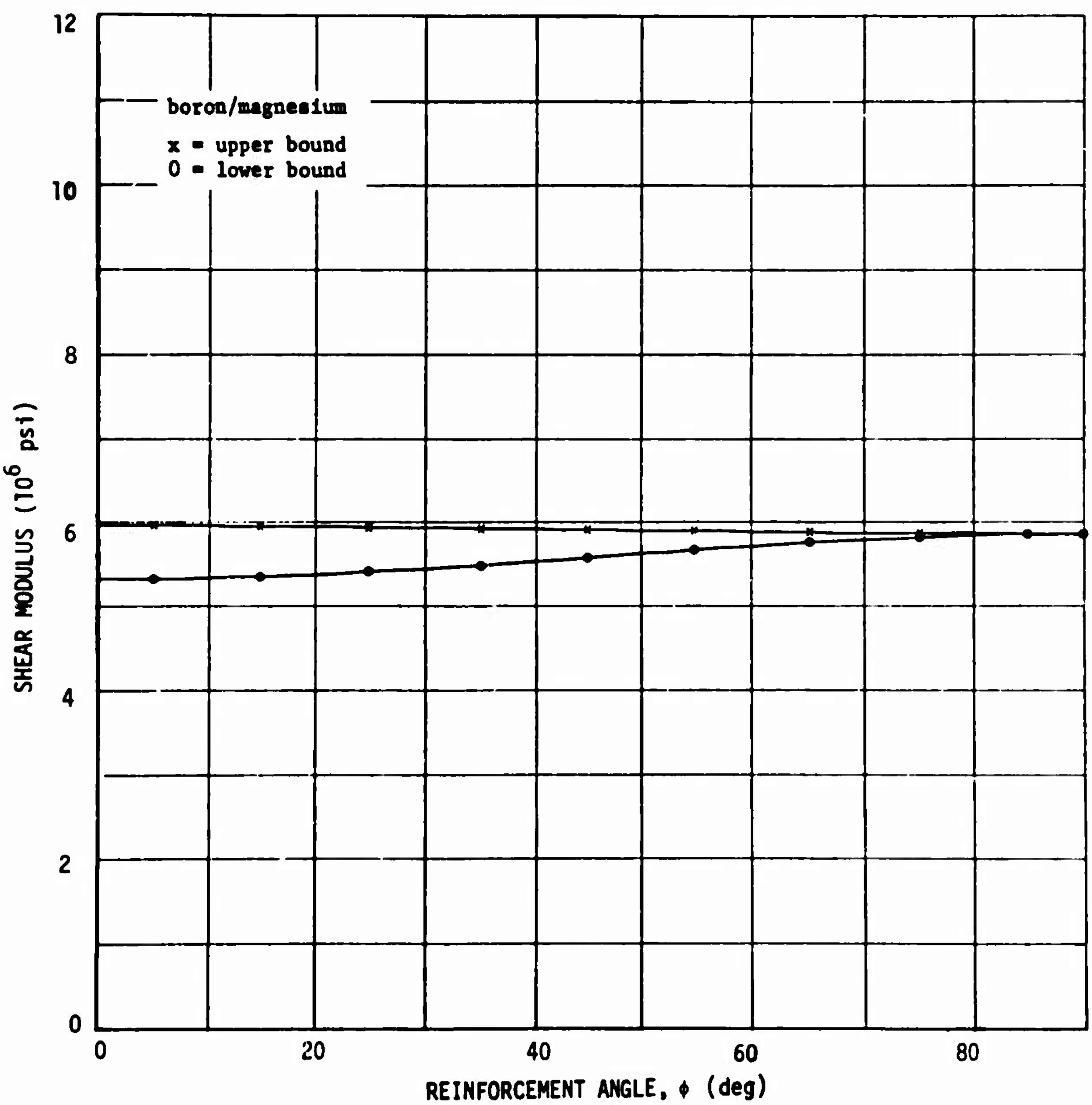

Figure 32. Shear Modulus G_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.5

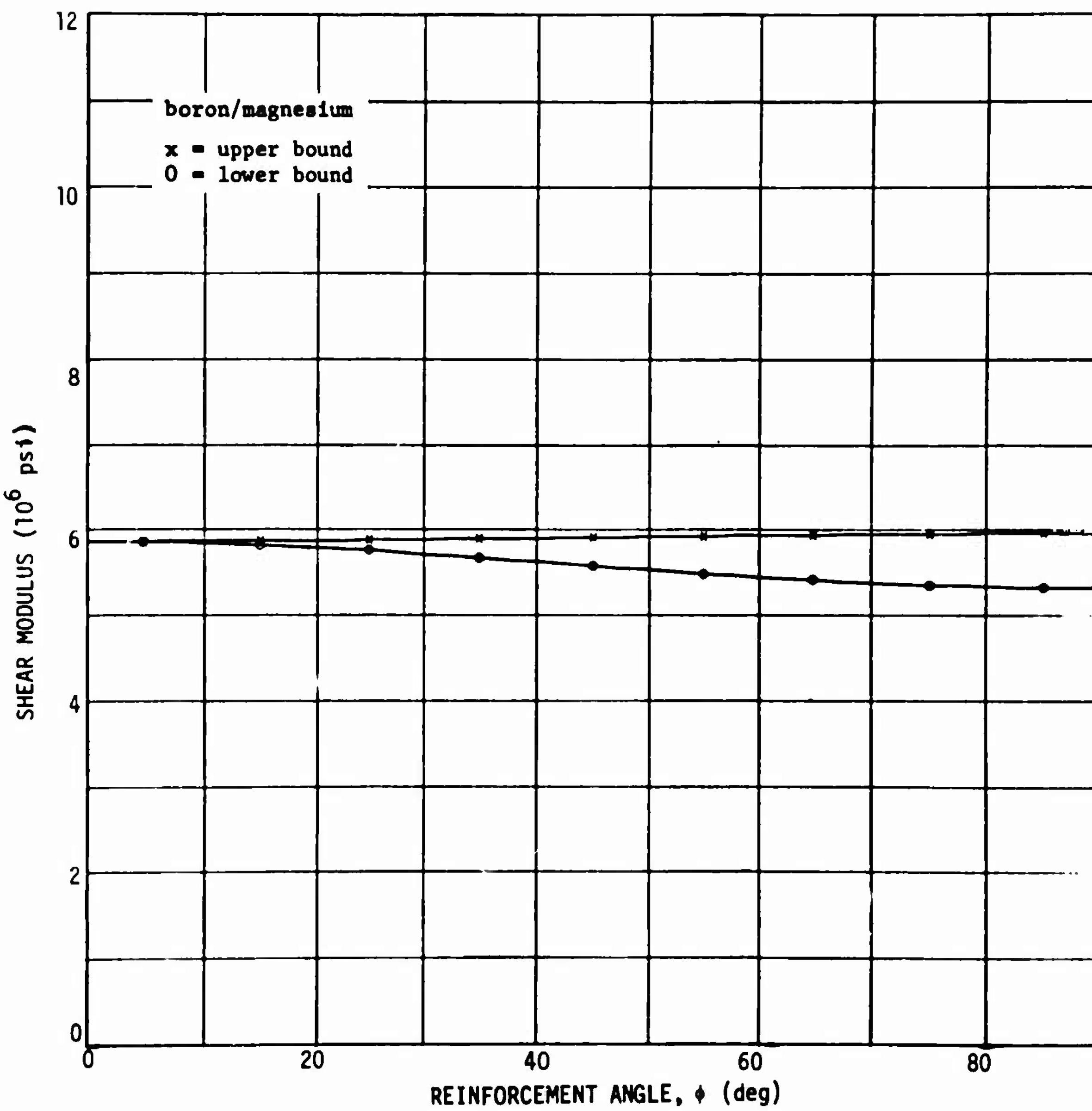

Figure 33. Shear Modulus G^*_{12} vs. Reinforcement Angle
Volume Fraction = 0.5

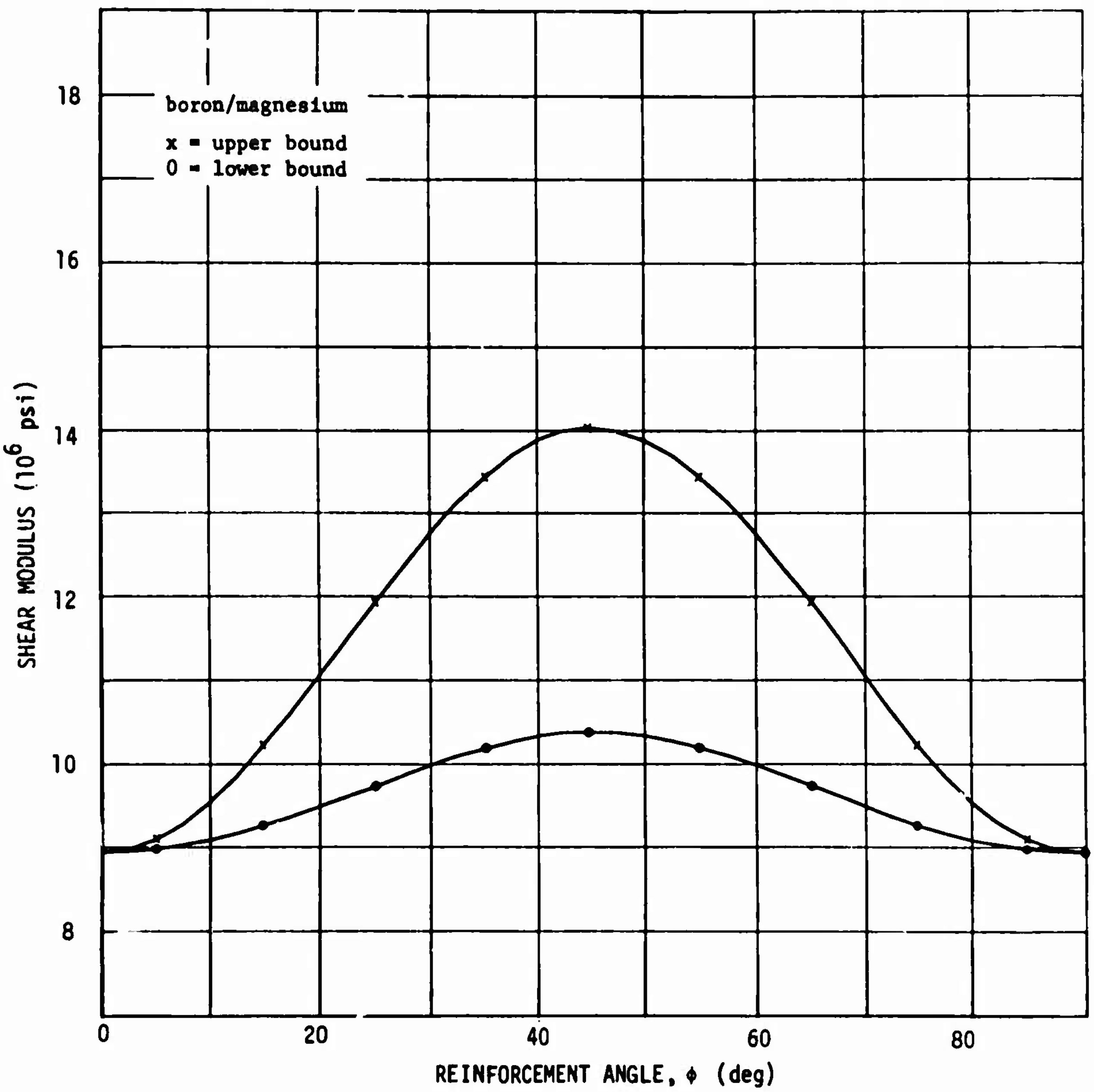

Figure 34. Shear Modulus G^*_{31} vs. Reinforcement Angle
Volume Fraction = 0.7

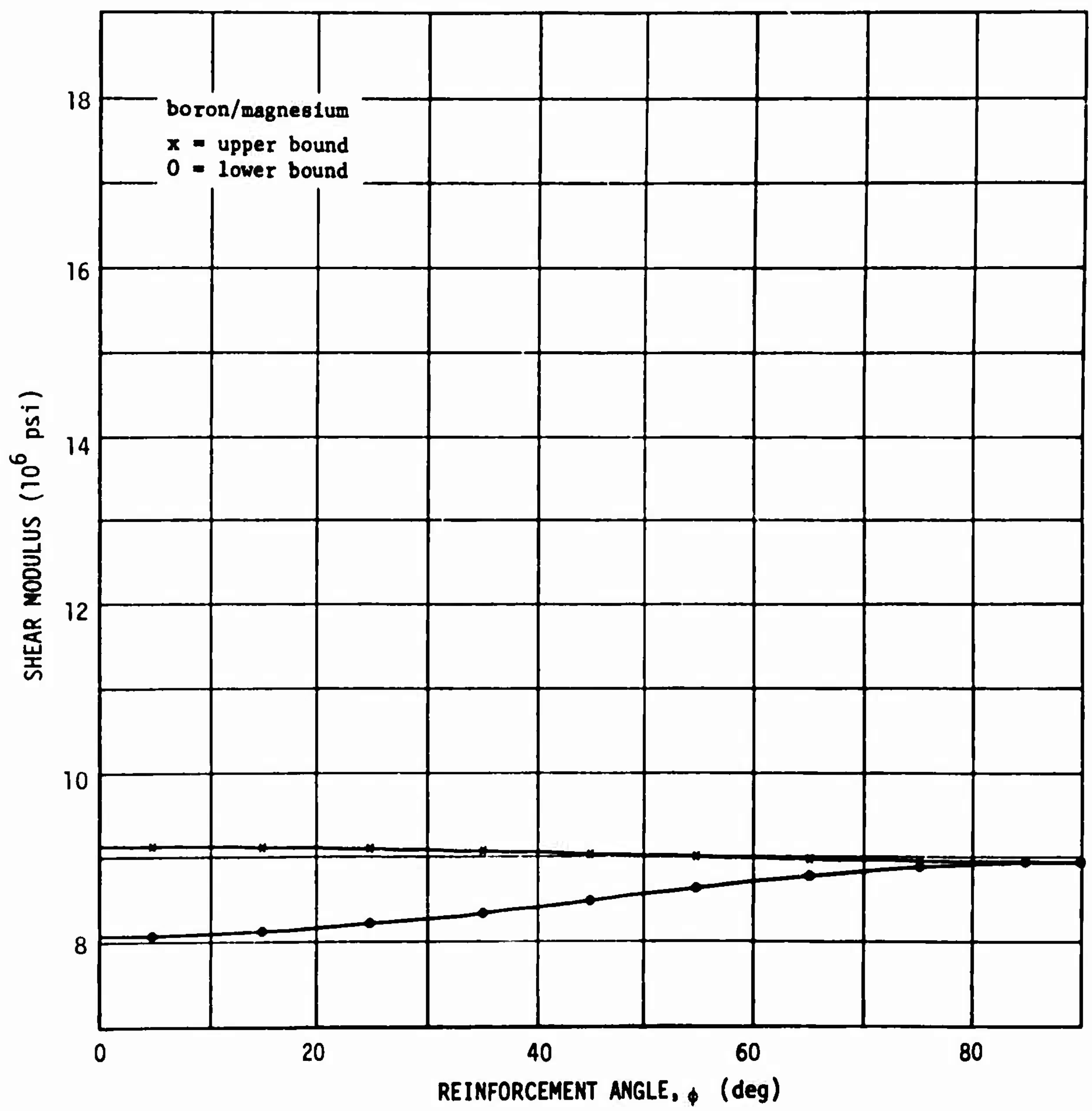

Figure 35. Shear Modulus G_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.7

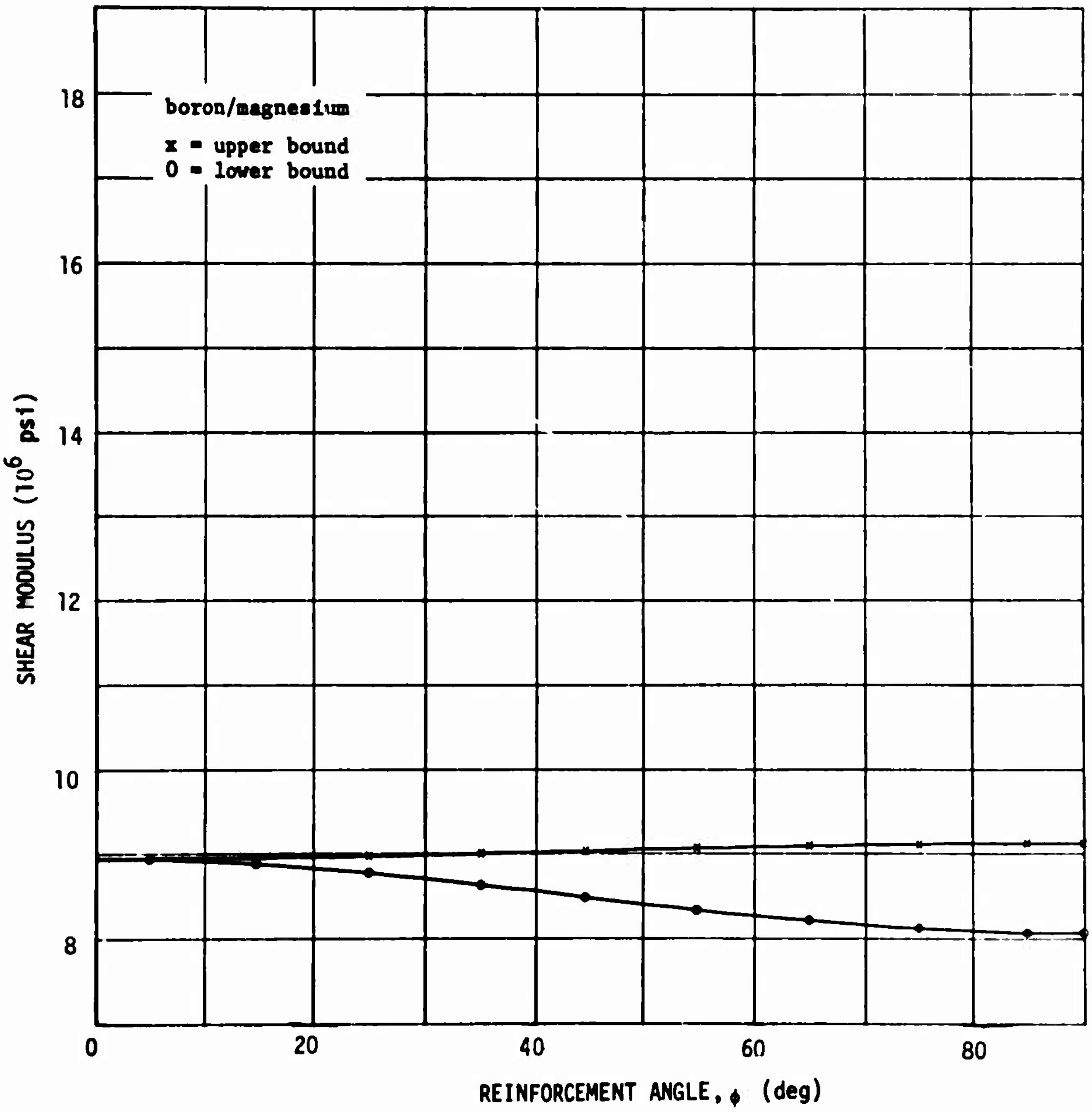

Figure 36. Shear Modulus G_{12}^* vs. Reinforcement Angle
Volume Fraction = 0.7

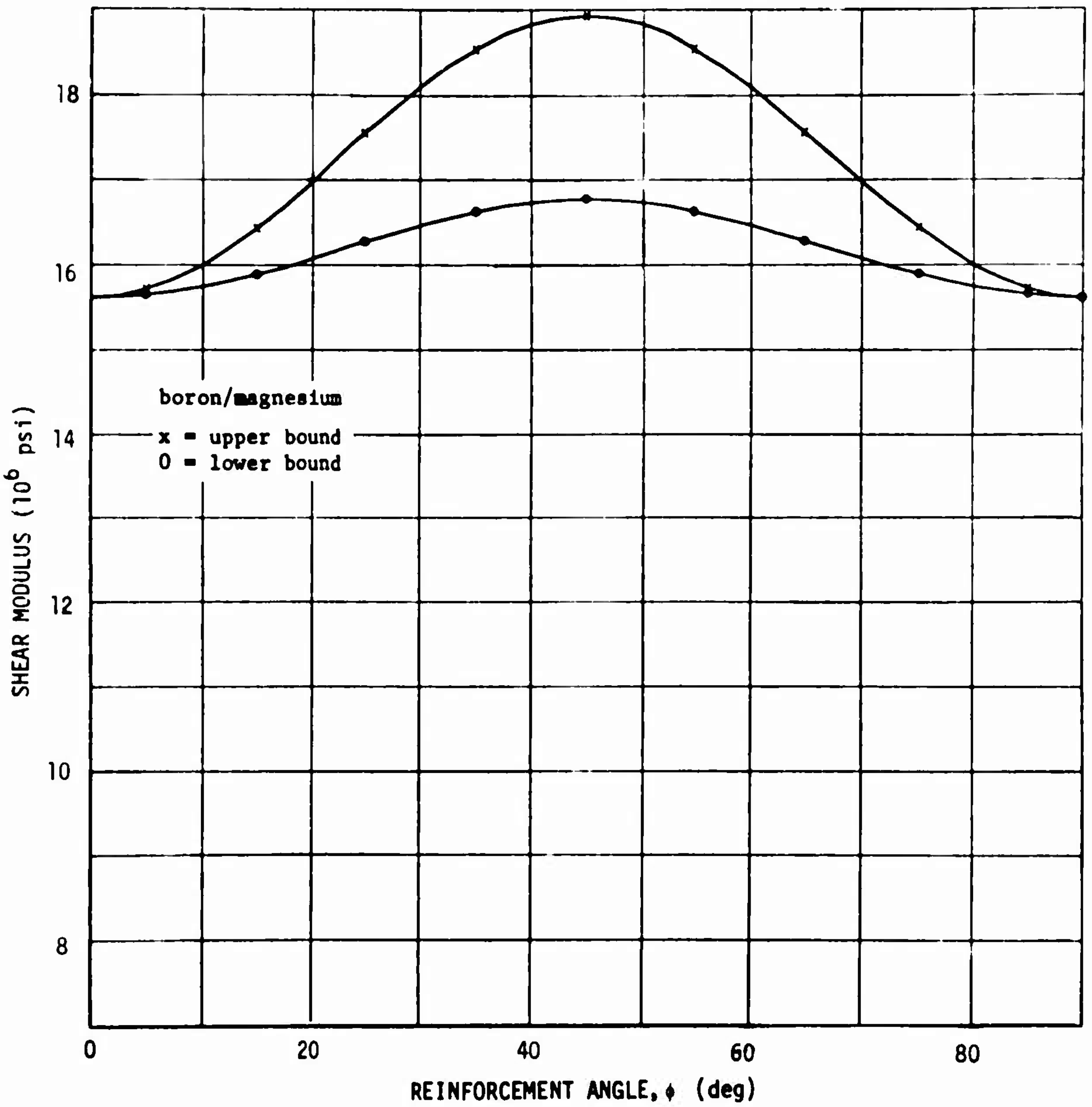

Figure 37. Shear Modulus G^*_{31} vs. Reinforcement Angle
Volume Fraction = 0.9

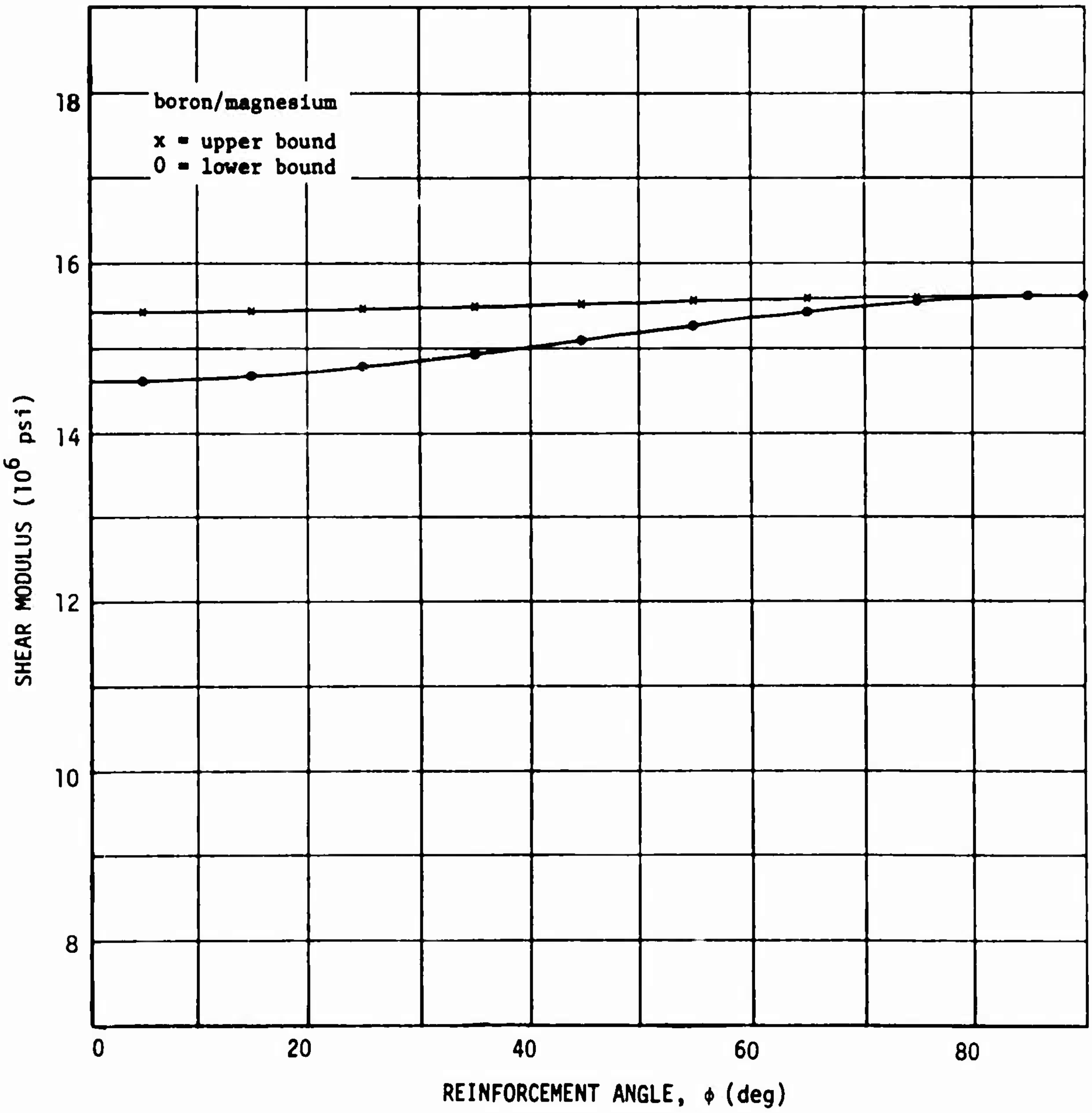

Figure 38. Shear Modulus G_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.9

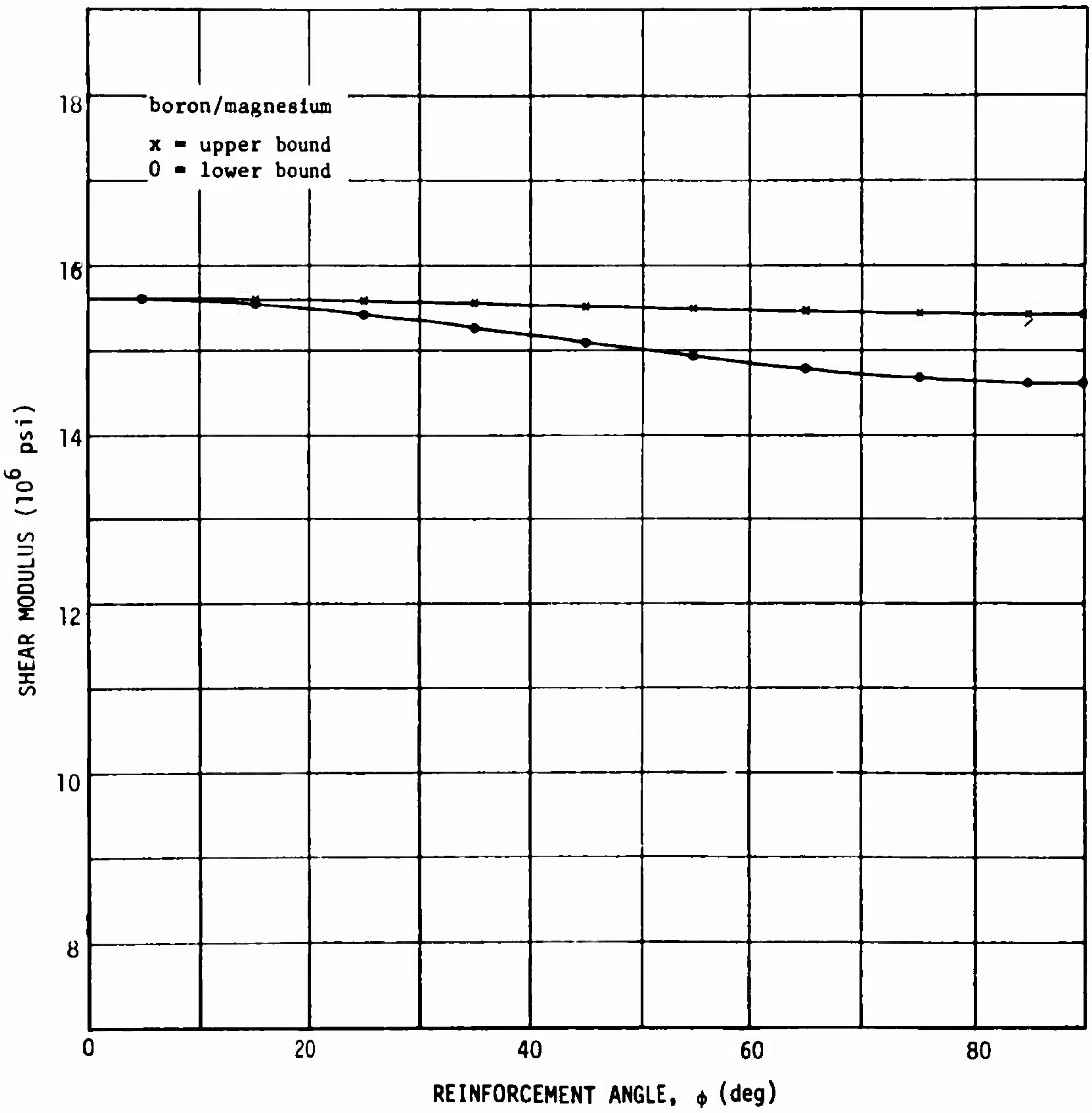

Figure 39. Shear Modulus G^*_{12} vs. Reinforcement Angle
Volume Fraction = 0.9

10. COMPUTATION OF NORMAL EFFECTIVE ELASTIC MODULI FOR LAYER MODEL

The normal effective elastic moduli of the composite are here defined as the six moduli $C^*_{ij} = C^*_{ji}$, i, j = 1,2,3 which relate the normal average stresses to the normal average strains; see (10). It will be shown here that if the composite is described by the layer model with transversely isotropic layers, then the normal C^*_{ij} can be exactly computed in terms of the layer moduli.

Let a specimen of the composite, as shown in Figure 1, be subjected to the boundary displacements

$$
\begin{aligned}
u_1(S) &= \varepsilon_{11}\, x_1, && \text{(a)}\\
u_2(S) &= \varepsilon_{22}\, x_2, && \text{(b)}\\
u_3(S) &= \varepsilon_{33}\, x_3, && \text{(c)}
\end{aligned}
\tag{143}
$$

where ε_{11}, ε_{22}, and ε_{33} are constant. It then follows from the general result (3) that the average strains are

$$
\begin{aligned}
\bar{\varepsilon}_{11} &= \varepsilon_{11}, && \text{(a)}\\
\bar{\varepsilon}_{22} &= \varepsilon_{22}, && \text{(b)}\\
\bar{\varepsilon}_{33} &= \varepsilon_{33}, && \text{(c)}\\
\bar{\varepsilon}_{12} &= \bar{\varepsilon}_{23} = \bar{\varepsilon}_{31} = 0. && \text{(d)}
\end{aligned}
\tag{144}
$$

The average stresses are then related to the average strains by (10). We shall show here that in the present case the average stresses can actually be computed. These then lead via (10) to expressions for the C^*_{ij} appearing in (10).

We look for an exact elasticity solution of the layered material under boundary conditions (143). We make the guess that the internal displacements <u>throughout</u> all layers are of the form (143). Thus,

$$
\begin{aligned}
u_1(\underline{x}) &= \varepsilon_{11}\, x_1, && \text{(a)}\\
u_2(\underline{x}) &= \varepsilon_{22}\, x_2, && \text{(b)}\\
u_3(\underline{x}) &= \varepsilon_{33}\, x_3. && \text{(c)}
\end{aligned}
\tag{145}
$$

For (145) to be an exact elasticity solution of the layered material, it must satisfy the following conditions:

(1) The boundary conditions (143).
(2) The differential equations of elasticity theory throughout all layers.
(3) Displacement and traction continuity at layer interfaces.

Now (143) is obviously satisfied, since the choice (145) was made for that purpose.

The differential equations of elasticity of the layers are <u>second-order</u> partial differential equations. Since the equations (145) are linear, they satisfy these equations trivially.

Displacement continuity at layer interfaces is obviously satisfied, since (145) is by choice the same continuous field in all the layers. Thus, only traction continuity at layer interfaces remains to be investigated. Since the normal to the layer interface is in the x_2 direction, Figure 1, the components of the traction at the interface are σ_{21}, σ_{22}, and σ_{23}. Thus, we must have

$$\left.\begin{matrix}\sigma_{21}\\ \sigma_{22}\\ \sigma_{23}\end{matrix}\right\}\ \text{continuous at layer interfaces} \tag{146}$$

This will now be investigated.

First, compute the strains ε'_{ij} and ε''_{ij} in the layers. The strains associated with (145) are

$$[\varepsilon_{ij}] = \begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix} . \tag{147}$$

Using the transformations (70) and (72), we find

$$\begin{aligned}
\varepsilon'_{11} &= \varepsilon_{11}\cos^2\phi + \varepsilon_{33}\sin^2\phi &&= \varepsilon''_{11} \,, && \text{(a)} \\
\varepsilon'_{22} &= \varepsilon_{22} &&= \varepsilon''_{22} \,, && \text{(b)} \\
\varepsilon'_{33} &= \varepsilon_{11}\sin^2\phi + \varepsilon_{33}\cos^2\phi &&= \varepsilon''_{33} \,, && \text{(c)} \\
\varepsilon'_{13} &= \left(\varepsilon_{33} - \varepsilon_{11}\right)\sin\phi\cos\phi &&= -\varepsilon''_{13} \,, && \text{(d)} \\
\varepsilon'_{12} &= 0 &&= \varepsilon''_{12} \,, && \text{(e)} \\
\varepsilon'_{23} &= 0 &&= \varepsilon''_{23} \,. && \text{(f)} \qquad (148)
\end{aligned}$$

The stresses σ'_{ij} in the primed layers, Figure 2a, are given by (42). Similarly, the stresses σ''_{ij} in the double primed layers, Figure 2b, are given by (42) with ε'_{ij} replaced by ε''_{ij}. Using (148), we find

$$\begin{aligned}
\sigma'_{11} &= \sigma''_{11}, && \text{(a)} \\
\sigma'_{22} &= \sigma''_{22}, && \text{(b)} \\
\sigma'_{33} &= \sigma''_{33}, && \text{(c)} \\
\sigma'_{12} &= \sigma''_{12} = 0, && \text{(d)} \\
\sigma'_{23} &= \sigma''_{23} = 0, && \text{(e)} \\
\sigma'_{13} &= -\sigma''_{13}. && \text{(f)} \qquad (150)
\end{aligned}$$

In order to investigate the continuity conditions (146), it is necessary to transform the layer stresses (149) back to the composite coordinate system x_1, x_2, x_3. For this purpose, we use the transformations (71) and (73). As has been done previously, we denote the left sides of (71) by $'\sigma_{ij}$ and the left sides of (73) by $''\sigma_{ij}$. This notation means stresses in layers referred to the x_1, x_2, x_3 system of axes expressed in terms of stress components referred to the x'_1, x'_2, x'_3 or x''_1, x''_2, x''_3 systems, respectively.

We now have, in view of (149),

$$\begin{aligned}
{}'\sigma_{21} &= {}''\sigma_{21} = 0\,, && \text{(a)} \\
{}'\sigma_{22} &= \sigma'_{22} = \sigma''_{22} = {}''\sigma_{22}, && \text{(b)} \\
{}'\sigma_{23} &= {}''\sigma_{23} = 0. && \text{(c)} \qquad (150)
\end{aligned}$$

Therefore, the continuity conditions (146) are satisfied. We thus conclude that the displacement field (145) is an exact elasticity solution for the layered composite.

We shall now need the average stresses $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$, and $\bar{\sigma}_{33}$. For this purpose, we first compute the uniform layer stresses ${}'\sigma_{11}$, ${}'\sigma_{22}$, ${}'\sigma_{33}$ and ${}''\sigma_{11}$, ${}''\sigma_{22}$, ${}''\sigma_{33}$. From (71 a,b,c) and (149), we find

$$
\begin{aligned}
{}'\sigma_{11} &= \sigma'_{11}\cos^2\phi + \sigma'_{33}\sin^2\phi - 2\sigma'_{13}\sin\phi\cos\phi, && \text{(a)}\\
{}'\sigma_{22} &= \sigma'_{22}, && \text{(b)}\\
{}'\sigma_{33} &= \sigma'_{11}\sin^2\phi + \sigma'_{33}\cos^2\phi + 2\sigma'_{13}\sin\phi\cos\phi. && \text{(c)} \qquad (151)
\end{aligned}
$$

Similarly, from (73 a,b,d) and (149), we have

$$
\begin{aligned}
{}''\sigma_{11} &= \sigma''_{11}\cos^2\phi + \sigma''_{33}\sin^2\phi + 2\sigma''_{13}\sin\phi\cos\phi, && \text{(a)}\\
{}''\sigma_{22} &= \sigma''_{22}, && \text{(b)}\\
{}''\sigma_{33} &= \sigma''_{11}\sin^2\phi + \sigma''_{33}\cos^2\phi - 2\sigma''_{13}\sin\phi\cos\phi. && \text{(c)} \qquad (152)
\end{aligned}
$$

In view of (149), it is seen from (151) and (152) that

$$
\begin{aligned}
{}'\sigma_{11} &= {}''\sigma_{11}, && \text{(a)}\\
{}'\sigma_{22} &= {}''\sigma_{22}, && \text{(b)}\\
{}'\sigma_{33} &= {}''\sigma_{33}, && \text{(c)} \qquad (153)
\end{aligned}
$$

where (153b) has already been found in connection with the continuity conditions.

Since the stresses (153) are uniform and since both kinds of layers are of equal volumes, we have

$$
\begin{aligned}
\bar{\sigma}_{11} &= {}'\sigma_{11}, && \text{(a)}\\
\bar{\sigma}_{22} &= {}'\sigma_{22}, && \text{(b)}\\
\bar{\sigma}_{33} &= {}'\sigma_{33}. && \text{(c)} \qquad (154)
\end{aligned}
$$

We first compute the layer stresses σ'_{ij} in terms of ε_{11}, ε_{22}, ε_{33}. To do this, (148) is used in (42), yielding

$$\sigma'_{11} = \left(C'_{11}\cos^2\phi + C'_{12}\sin^2\phi\right)\varepsilon_{11} + C'_{12}\,\varepsilon_{22}$$
$$+ \left(C'_{11}\sin^2\phi + C'_{12}\cos^2\phi\right)\varepsilon_{33} \quad , \qquad \text{(a)}$$
$$\sigma'_{22} = \left(C'_{12}\cos^2\phi + C'_{23}\sin^2\phi\right)\varepsilon_{11} + C'_{22}\,\varepsilon_{22}$$
$$+ \left(C'_{12}\sin^2\phi + C'_{23}\cos^2\phi\right)\varepsilon_{33} \quad , \qquad \text{(b)}$$
$$\sigma'_{33} = \left(C'_{12}\cos^2\phi + C'_{22}\sin^2\phi\right)\varepsilon_{11} + C'_{23}\,\varepsilon_{22}$$
$$+ \left(C'_{12}\sin^2\phi + C'_{22}\cos^2\phi\right)\varepsilon_{33} \quad , \qquad \text{(c)}$$
$$\sigma'_{13} = 2C'_{44}\left(\varepsilon_{33} - \varepsilon_{11}\right)\sin\phi\cos\phi \quad . \qquad \text{(d)} \qquad (155)$$

Now (155) is substituted into (151), which in view of (154) are also the average stresses. After rearrangement, we have

$$\bar{\sigma}_{11} = \Big[C'_{11}\cos^4\phi + C'_{22}\sin^4\phi + 2C'_{12}\cos^2\phi\sin^2\phi$$
$$+ 4C'_{44}\sin^2\phi\cos^2\phi\Big]\varepsilon_{11} + \Big[C'_{12}\cos^2\phi\; C'_{23}\sin^2\phi\Big]\varepsilon_{22}$$
$$+ \Big[\left(C'_{11} + C'_{22}\right)\sin^2\phi\cos^2\phi + C'_{12}\left(\sin^4\phi + \cos^4\phi\right)$$
$$- 4C'_{44}\sin^2\phi\cos^2\phi\Big]\varepsilon_{33}, \qquad \text{(a)}$$

$$\bar{\sigma}_{22} = \Big[C'_{12}\cos^2\phi + C'_{23}\sin^2\phi\Big]\varepsilon_{11} + C'_{22}\,\varepsilon_{22}$$
$$+ \Big[C'_{12}\sin^2\phi + C'_{23}\cos^2\phi\Big]\varepsilon_{33}, \qquad \text{(b)}$$

$$\bar{\sigma}_{33} = \Big[\left(C'_{11} + C'_{22}\right)\sin^2\phi\cos^2\phi + C'_{12}\left(\sin^4\phi + \cos^4\phi\right)$$
$$- 4C'_{44}\sin^2\phi\cos^2\phi\Big]\varepsilon_{11} + \Big[C'_{12}\sin^2\phi + C'_{23}\cos^2\phi\Big]\varepsilon_{22}$$
$$+ \Big[C'_{11}\sin^4\phi + C'_{22}\cos^4\phi + 2C'_{12}\cos^2\phi\sin^2\phi$$
$$+ 4C'_{44}\cos^2\phi\sin^2\phi\Big]\varepsilon_{33}. \qquad \text{(c)} \qquad (156)$$

Comparing (156) with (10), we see that the effective elastic moduli are

$$C^*_{11} = C'_{11}\cos^4\phi + C'_{22}\sin^4\phi + 2C'_{12}\cos^2\phi\sin^2\phi$$
$$+ 4C'_{44}\cos^2\phi\sin^2\phi, \qquad (157)$$

$$C^*_{12} = C^*_{21} = C'_{12}\cos^2\phi + C'_{23}\sin^2\phi, \qquad (158)$$

$$C^*_{13} = C^*_{31} = \left(C'_{11} + C'_{22}\right)\cos^2\phi\sin^2\phi + C'_{12}\left(\sin^4\phi + \cos^4\phi\right),$$
$$- 4C'_{44}\cos^2\phi\sin^2\phi, \qquad (159)$$

$$C^*_{22} = C'_{22}, \qquad (160)$$

$$C^*_{23} = C^*_{32} = C'_{12}\sin^2\phi + C'_{23}\cos^2\phi, \qquad (161)$$

$$C^*_{33} = C'_{11}\sin^4\phi + C'_{22}\cos^4\phi + 2C'_{12}\cos^2\phi\sin^2\phi$$
$$+ 4C'_{44}\cos^2\phi\sin^2\phi. \qquad (162)$$

If the elastic moduli of the layers are known from experiment or theory, then (157) through (162) enable one to compute the six effective elastic moduli in (10). However, experimental determination of the C'_{ij} is difficult. Therefore, it is best to proceed on the basis of theoretical knowledge about the C'_{ij}. It will be shown in the next section how (157) through (162) can be used to bound the moduli C^*_{ij}.

11. BOUNDS FOR NORMAL EFFECTIVE ELASTIC MODULI FOR COMPOSITE CYLINDER MODEL

At the present time, no geometrical model of a uniaxially fiber reinforced material for which all five elastic moduli can be rigorously determined is available.

The composite cylinder assemblage model of Hashin and Rosen [1] comes closest to such a model, since it permits rigorous computation in closed form of four of the five elastic moduli; see Section 5. Therefore, we shall assume that the layers are described by this model, and we shall now proceed to exploit the theory developed in Section 10 for this case.

First, the layer moduli appearing in (157) through (162) are rewritten by use of (44) through (48). Thus,

$$C'_{11} = E_a + 4\, K_t\, \nu_a^2 , \tag{163}$$

$$C'_{12} = 2\, K_t\, \nu_a , \tag{164}$$

$$C'_{22} = K_t + G_t , \tag{165}$$

$$C'_{23} = K_t - G_t , \tag{166}$$

$$C'_{44} = G_a . \tag{167}$$

Inserting these expressions into (157) through (162), we have

$$C^*_{11} = E_a \cos^4\phi + K_t\, (2\nu_a \cos^2\phi + \sin^2\phi)^2 \tag{168}$$

$$+ 4\, G_a \cos^2\phi \sin^2\phi + G_t \sin^4\phi , \tag{169}$$

$$C^*_{12} = K_t\, (2\nu_a \cos^2\phi + \sin^2\phi) - G_t \sin^2\phi , \tag{170}$$

$$C^*_{13} = E_a \cos^2\phi \sin^2\phi + K_t (2\nu_a \cos^2\phi + \sin^2\phi)(2\nu_a \sin^2\phi + \cos^2\phi)$$
$$- 4\, G_a \cos^2\phi \sin^2\phi + G_t \cos^2\phi \sin^2\phi , \tag{171}$$

$$C^*_{22} = K_t + G_t , \tag{172}$$

$$C^*_{23} = K_t\, (2\nu_a \sin^2\phi + \cos^2\phi) - G_t \cos^2\phi , \tag{173}$$

$$C^*_{33} = E_a \sin^4\phi + K_t\ (2\nu_a \sin^2\phi + \cos^2\phi)^2 + 4\ G_a \cos^2\phi \sin^2\phi$$
$$+ G_t \cos^4\phi\ . \qquad (174)$$

The following problem now arises: While the theory which has been given in [1] gives rigorous results in closed form for the moduli E_a, ν_a, K_t, and G_a, only lower and upper bounds for G_t are available; see (64) and (65). It is seen that G_t appears in every one of the Equations (168) through (174). Therefore, only the upper and lower bounds can be calculated for the moduli in these equations.

A rigorous procedure for bound construction for the effective moduli appearing on the left sides of (168) through (174) would consist of the following steps. Displacements of the form of (143) are applied as boundary conditions to the surfaces of all composite cylinders of which the layers are assumed to consist, see Figure 3. A solution can be obtained for a typical composite cylinder and, therefore, for each of the cylinders in the layers. The assemblage of displacement fields in all the cylinders then becomes an admissable field for the principal of minimum potential energy.

Then the sum of the strain energies in all cylinders subjected to (143) on their surfaces is larger than the composite strain energy, and an inequality of the general type (35) is obtained. The right-hand side of (35) in the present case is a quadratic form in which the variables are ε_{11}, ε_{22}, ε_{33} appearing in (143), and the coefficients are expressed in terms of the CCM moduli E_a, K_t, ν_a, G_a, and $G_t^{(+)}$ which is given by (65). The left-hand side of (35) now contains the unknown normal effective moduli of the composite written on the left-hand sides of (157) through (162).

An entirely similar procedure can be used with traction boundary conditions involving only normal stresses σ_{11}, σ_{22}, and σ_{33} with respect to the composite system of axes. The general procedure has been described in Section 4 and now leads to an inequality of type (41). The inequalities of type (35) and (41) have to be exploited to find the upper and lower bounds for C^*_{ij}. As has been mentioned in Sections 4 and 7, it is at present not known how to solve such inequalities. Therefore, we shall have to adopt a different procedure.

It will simply be assumed that if the layers are described by the CCM, we can replace the whole composite by one whose layers are homogeneous and have the uniaxial CCM elastic moduli. It is quite clear that the changes thus introduced into the effective elastic moduli of the CCM

biaxial composite will be insignificant, if each layer contains an appreciable number of fibers through its thickness. On the basis of this assumption, bounds for C^*_{ij} are obtained immediately by replacement of G_t in (168) through (174) by its lower or upper bounds, as given by (64) and (65). It is seen that in Equations (168), (170), (171), and (174), G_t appears with positive coefficients. Therefore, these expressions are a monotonically increasing function of G_t. Consequently, insertion of upper bound values for G_t results in upper bounds for the left sides, and insertion of lower bound values results in lower bounds for the left sides. In equations (169) and (173) G_t appears with negative coefficients, so these expressions monotonically decrease with increasing G_t. Therefore, upper bound values of G_t give lower bounds for the left sides, and lower bound values of G_t give upper bounds for the left sides.

As a result of these considerations, we can now write the following set of bounds for the effective moduli C^*_{ij} on the left side of (168) through (174).

$$C^{*(-)}_{11} \le C^*_{11} \le C^{*(+)}_{11} \,, \qquad \text{(a)}$$

$$C^{*(-)}_{11} = E_a \cos^4\phi + K_t \,(2\nu_a \cos^2\phi + \sin^2\phi)^2 + 4\, G_a \cos^2\phi \sin^2\phi + G^{(-)}_t \sin^4\phi \,, \qquad \text{(b)}$$

$$C^{*(+)}_{11} = E_a \cos^4\phi + K_t \,(2\nu_a \cos^2\phi + \sin^2\phi)^2 + 4\, G_a \cos^2\phi \sin^2\phi + G^{(+)}_t \sin^4\phi \,, \qquad \text{(c)} \quad (175)$$

$$C^{*(-)}_{12} \le C^*_{12} \le C^{*(+)}_{12} \,, \qquad \text{(a)}$$

$$C^{*(-)}_{12} = K_t \,(2\nu_a \cos^2\phi + \sin^2\phi) - G^{(+)}_t \sin^2\phi \,, \qquad \text{(b)}$$

$$C^{*(+)}_{12} = K_t \,(2\nu_a \cos^2\phi + \sin^2\phi) - G^{(-)}_t \sin^2\phi \,, \qquad \text{(c)} \quad (176)$$

$$C^{*(-)}_{13} \le C^*_{13} \le C^{*(+)}_{13} \,, \qquad \text{(a)}$$

$$C^{*(-)}_{13} = E_a \cos^2\phi \sin^2\phi + K_t \,(2\nu_a \cos^2\phi + \sin^2\phi)(2\nu_a \sin^2\phi + \cos^2\phi) - 4\, G_a \cos^2\phi \sin^2\phi + G^{(-)}_t \cos^2\phi \sin^2\phi \,, \qquad \text{(b)}$$

$$C^{*(+)}_{13} = E_a \cos^2\phi \sin^2\phi + K_t \,(2\nu_a \cos^2\phi + \sin^2\phi)(2\nu_a \sin^2\phi + \cos^2\phi) - 4\, G_a \cos^2\phi \sin^2\phi + G^{(+)}_t \cos^2\phi \sin^2\phi \,, \qquad \text{(c)} \quad (177)$$

$$C_{22}^{*(-)} \leq C_{22}^{*} \leq C_{22}^{*(+)}, \qquad \text{(a)}$$

$$C_{22}^{*(-)} = K_t + G_t^{(-)}, \qquad \text{(b)}$$

$$C_{22}^{*(+)} = K_t + G_t^{(+)}, \qquad \text{(c)} \qquad (178)$$

$$C_{23}^{*(-)} \leq C_{23}^{*} \leq C_{23}^{*(+)}, \qquad \text{(a)}$$

$$C_{23}^{*(-)} = K_t \, (2\nu_a \sin^2\phi + \cos^2\phi) - G_t^{(+)} \cos^2\phi \, , \qquad \text{(b)}$$

$$C_{23}^{*(+)} = K_t \, (2\nu_a \sin^2\phi + \cos^2\phi) - G_t^{(-)} \cos^2\phi \, , \qquad \text{(c)} \quad (179)$$

$$C_{33}^{*(-)} \leq C_{33}^{*} \leq C_{33}^{*(+)} \, , \qquad \text{(a)}$$

$$C_{33}^{*(-)} = E_a \sin^4\phi + K_t \, (2\nu_a \sin^2\phi + \cos^2\phi)^2 + 4 \, G_a \cos^2\phi \sin^2\phi + G_t^{(-)} \cos^4\phi \, , \qquad \text{(b)}$$

$$C_{33}^{*(+)} = E_a \sin^4\phi + K_t \, (2\nu_a \sin^2\phi + \cos^2\phi)^2 + 4 \, G_a \cos^2\phi \sin^2\phi + G_t^{(+)} \cos^4\phi \, . \qquad \text{(c)} \quad (180)$$

The bounds (175) through (180) have been numerically evaluated for fiber glass/epoxy and boron/magnesium composites whose constituent elastic properties have been listed in Section 9. The uniaxial layer composite moduli have again been computed on the basis of the results listed in Section 9.

Bounds for fiber glass/epoxy composites for varying angles of reinforcement and fiber volume fractions are given in Figures 40 through 57, while similar bounds for boron/magnesium composites are given in Figures 58 through 72.

It is seen that in a considerable number of cases, the bounds are so close that they are practically indistinguishable.

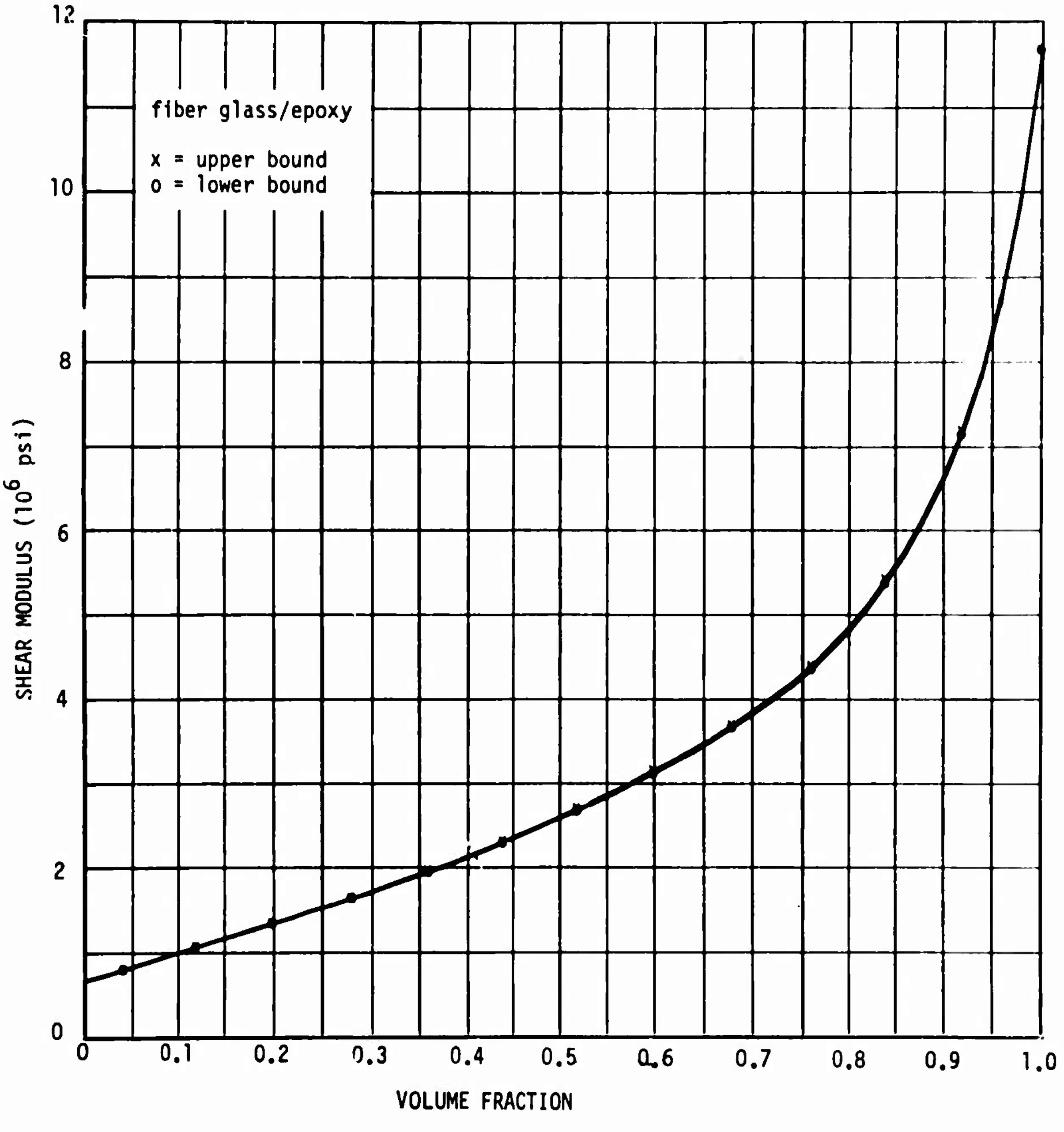

Figure 40. Effective Modulus C_{11}^* vs. Volume Fraction
ϕ = 45 degrees

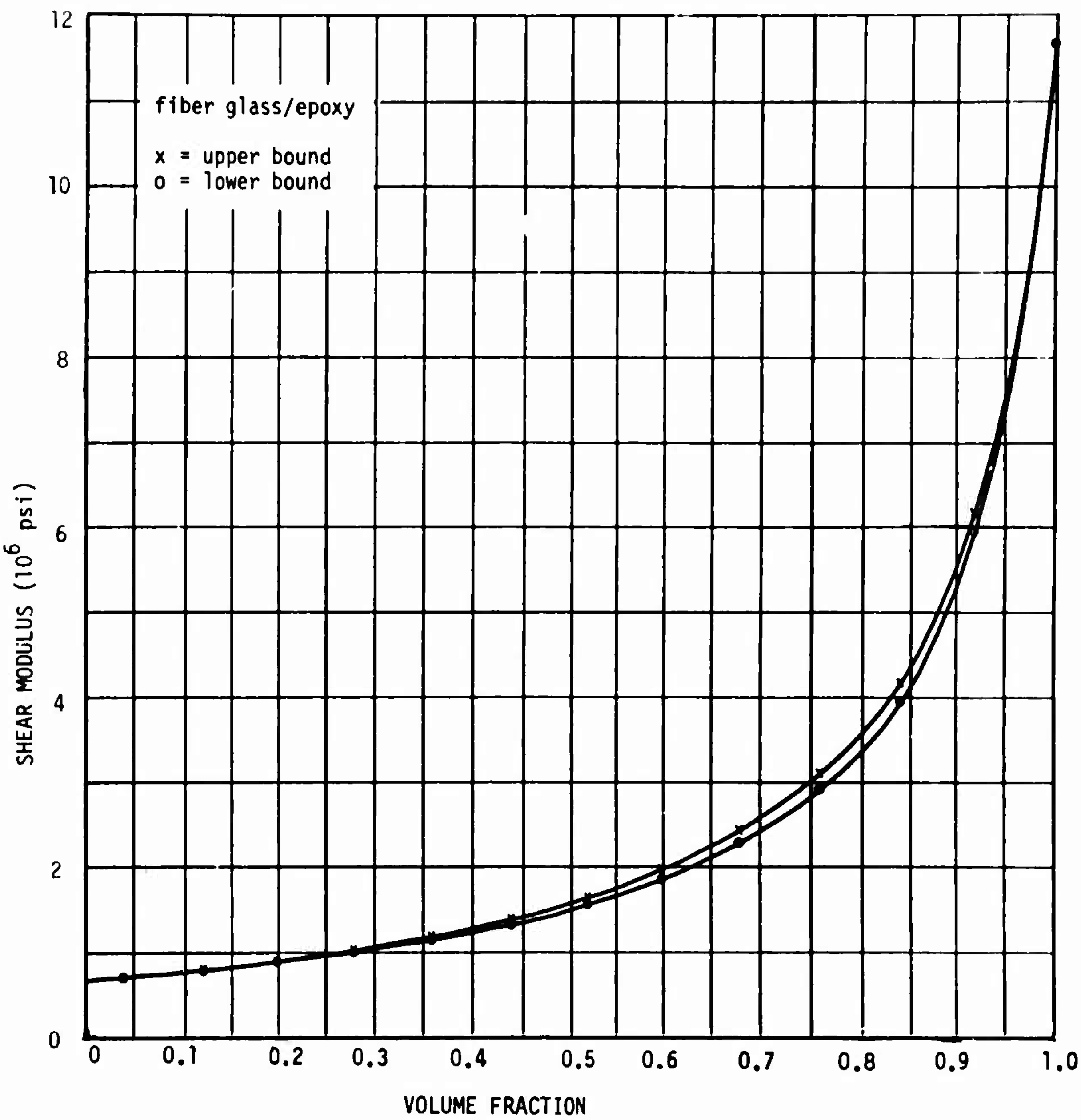

Figure 41. Effective Modulus C_{22}^* vs. Volume Fraction
ϕ = 45 degrees

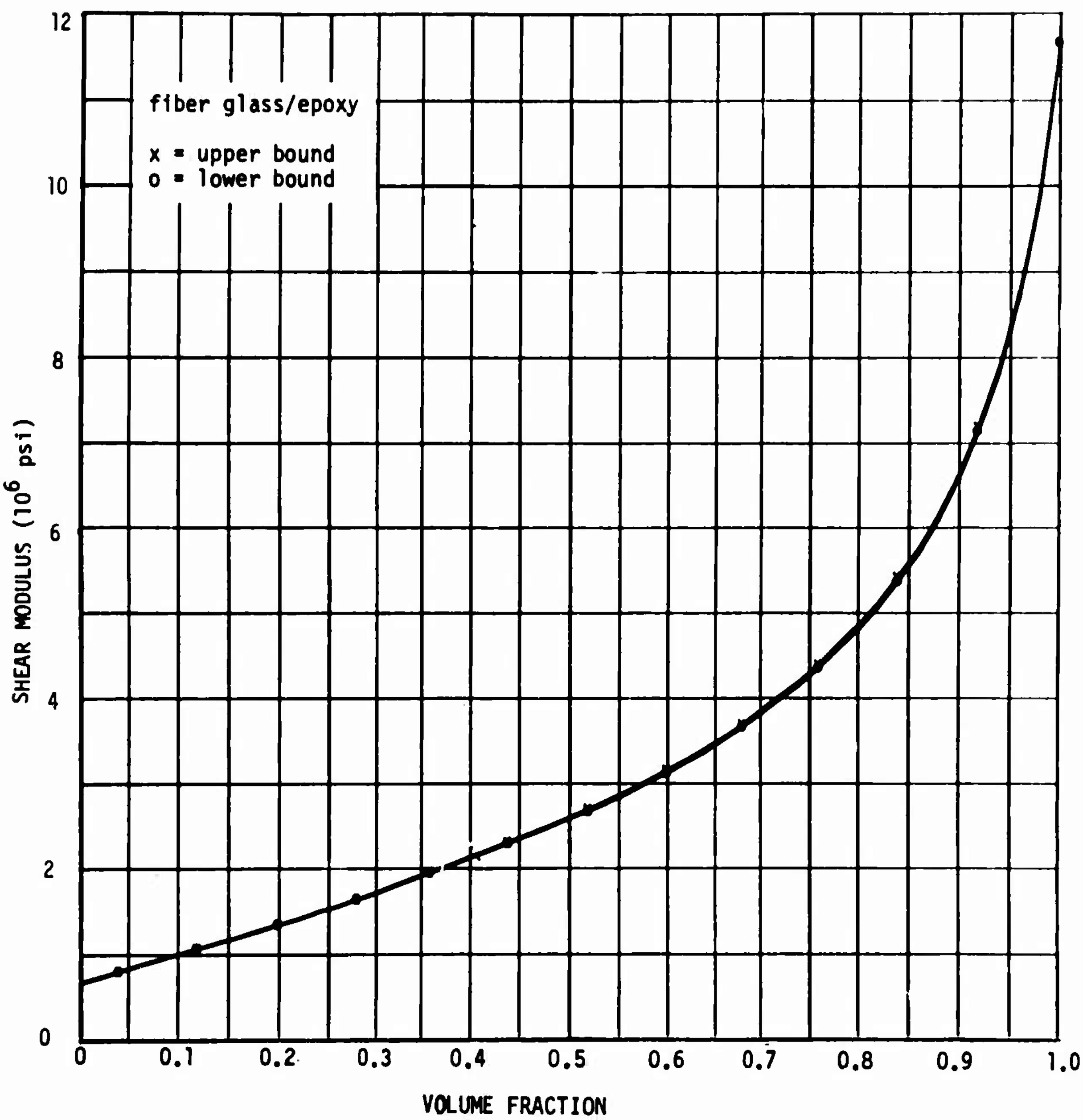

Figure 42. Effective Modulus C^*_{33} vs. Volume Fraction
ϕ = 45 degrees

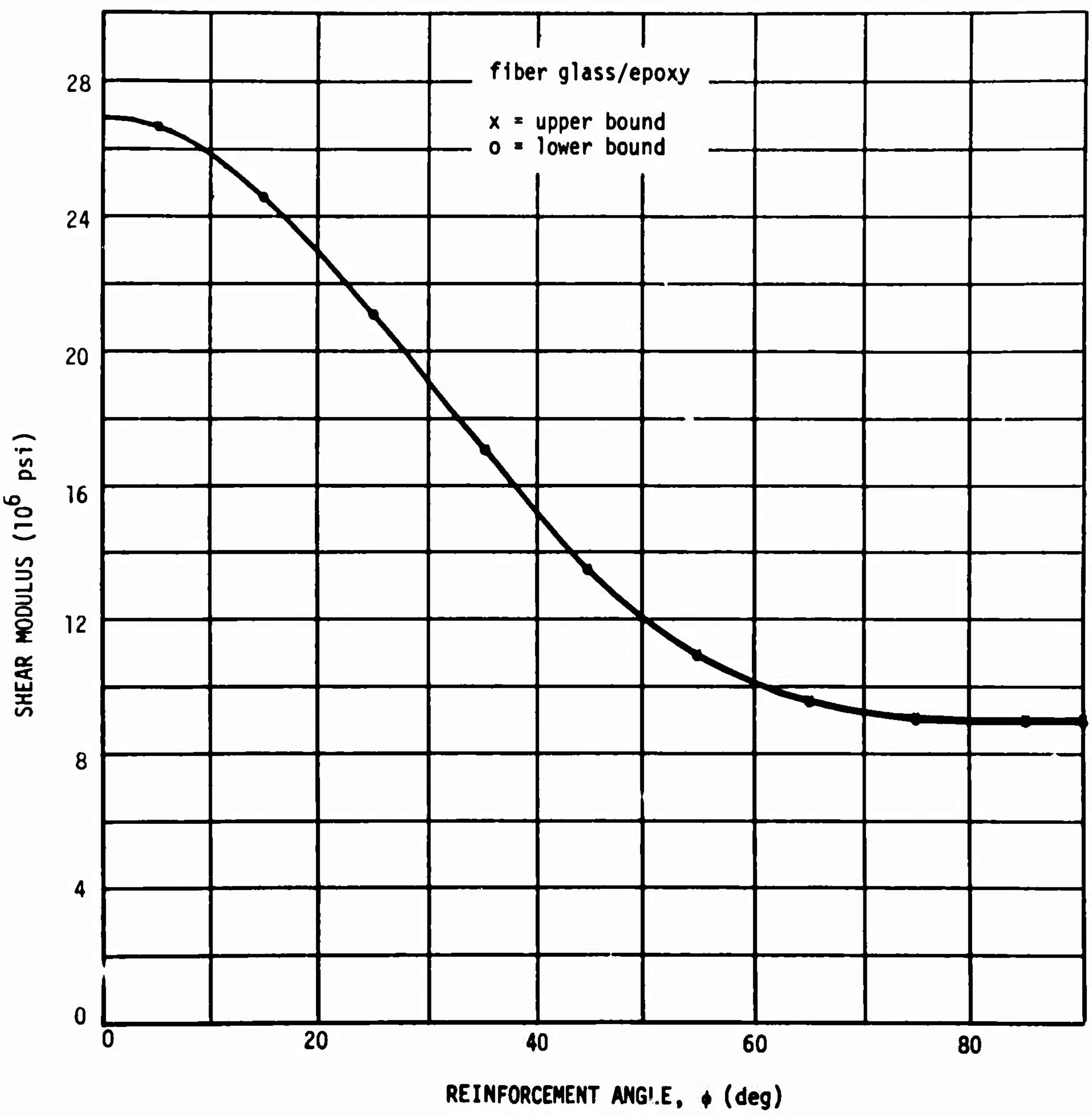

Figure 43. Effective Modulus C_{11}^* vs. Reinforcement Angle
Volume Fraction = 0.2

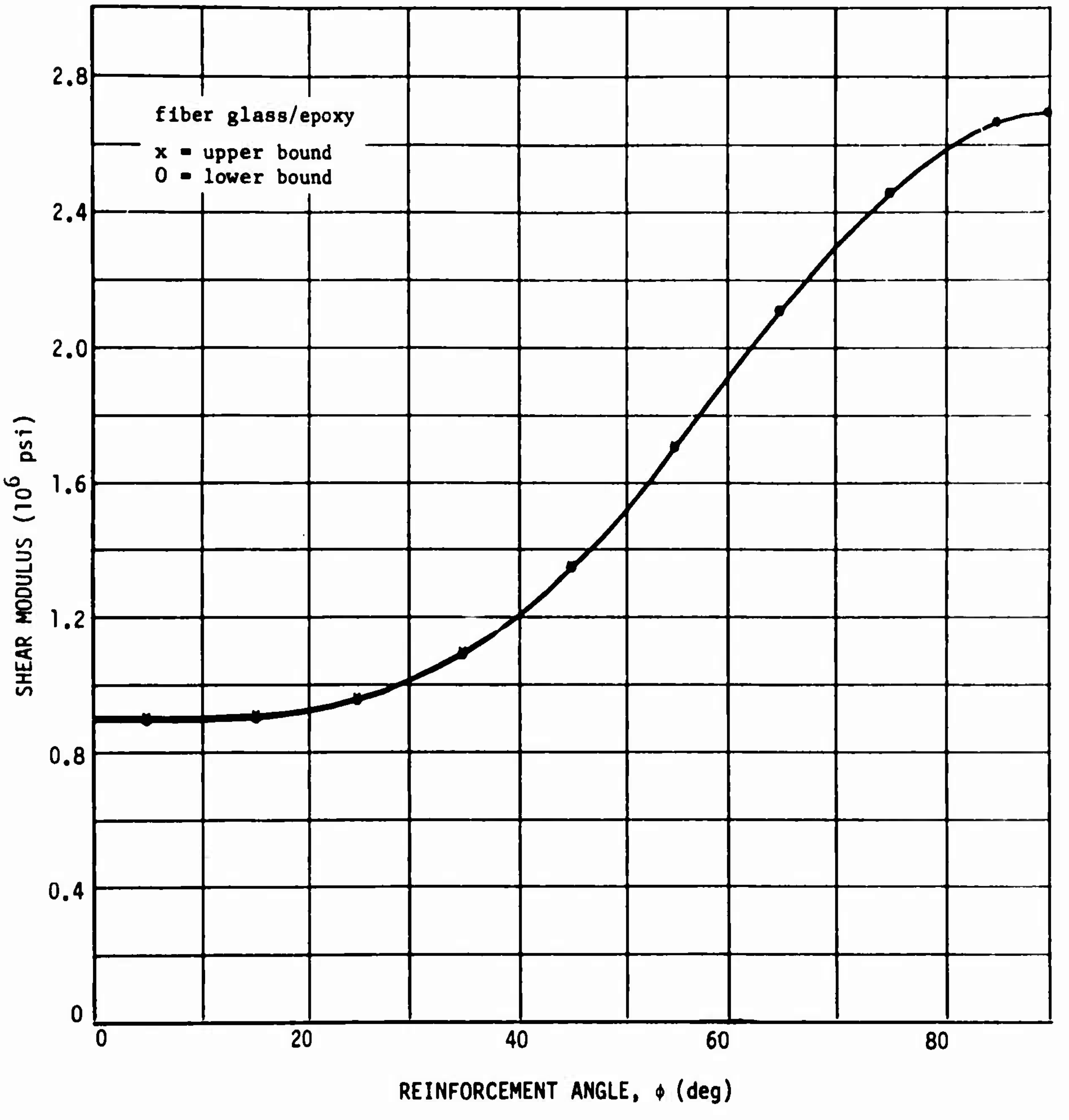

Figure 44. Effective Modulus C_{33}^* vs. Reinforcement Angle
Volume Fraction = 0.2

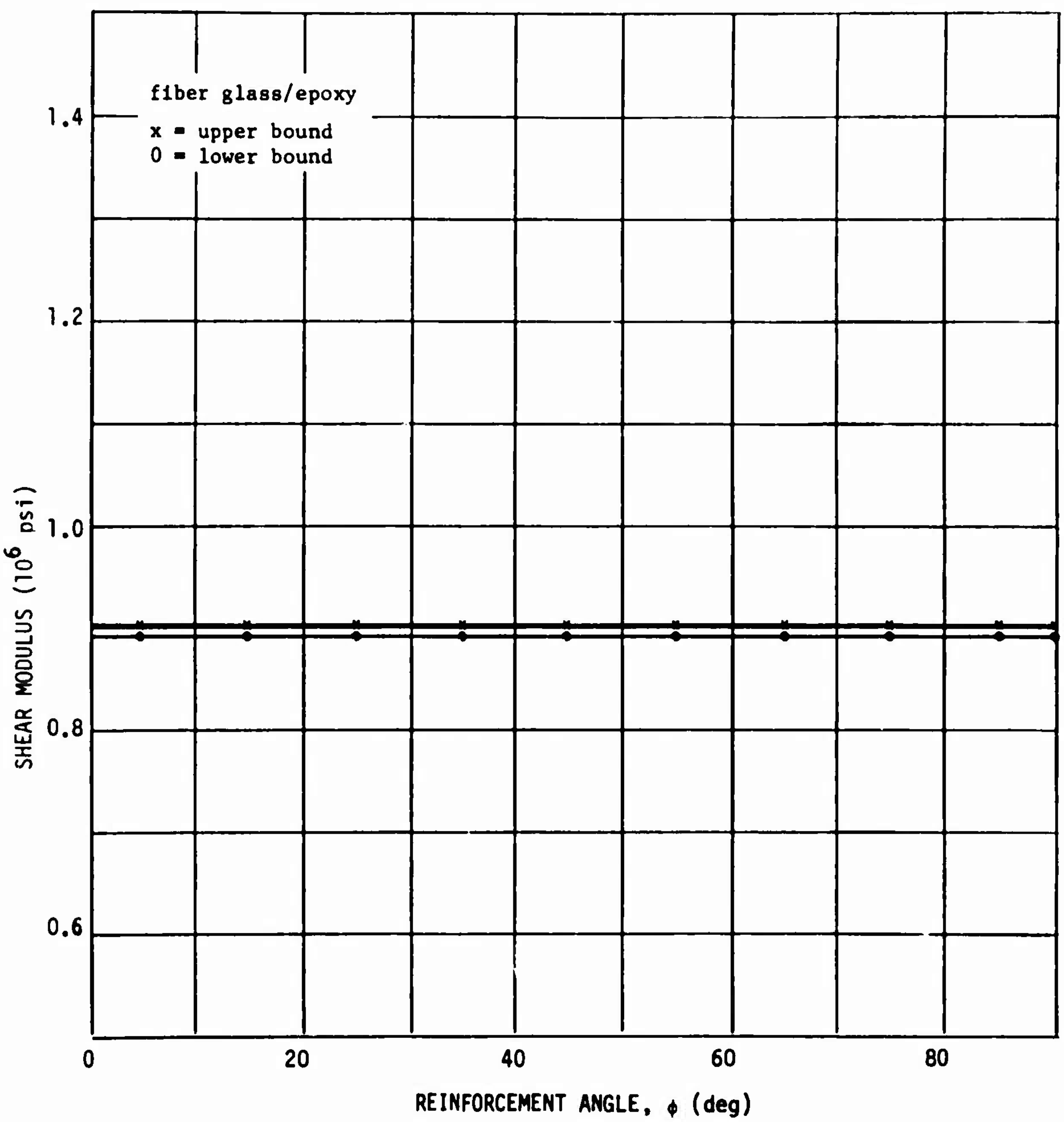

Figure 45. Effective Modulus C^*_{22} vs. Reinforcement Angle
Volume Fraction = 0.2

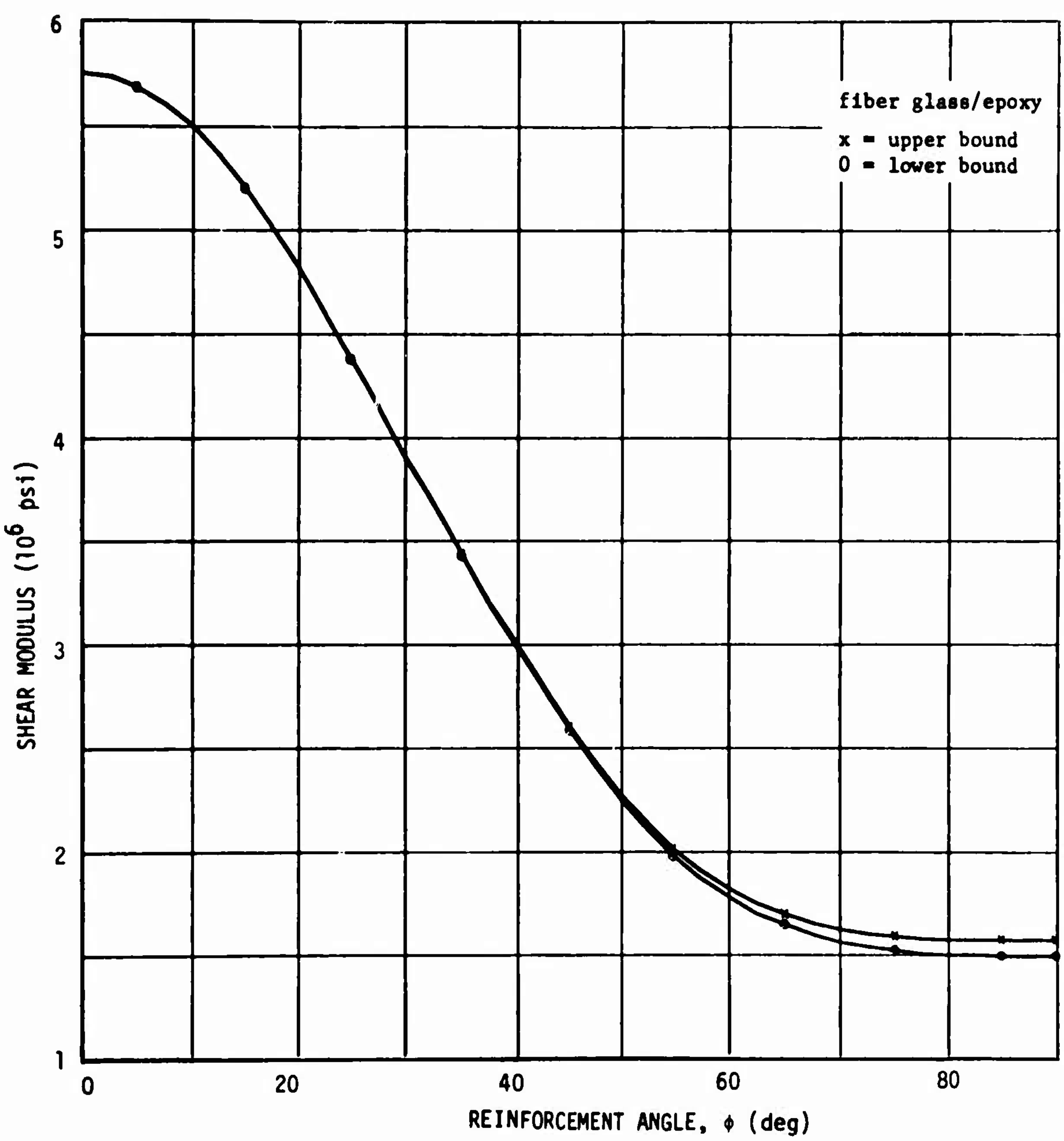

Figure 46. Effective Modulus C_{11}^* vs. Reinforcement Angle
Volume Fraction = 0.5

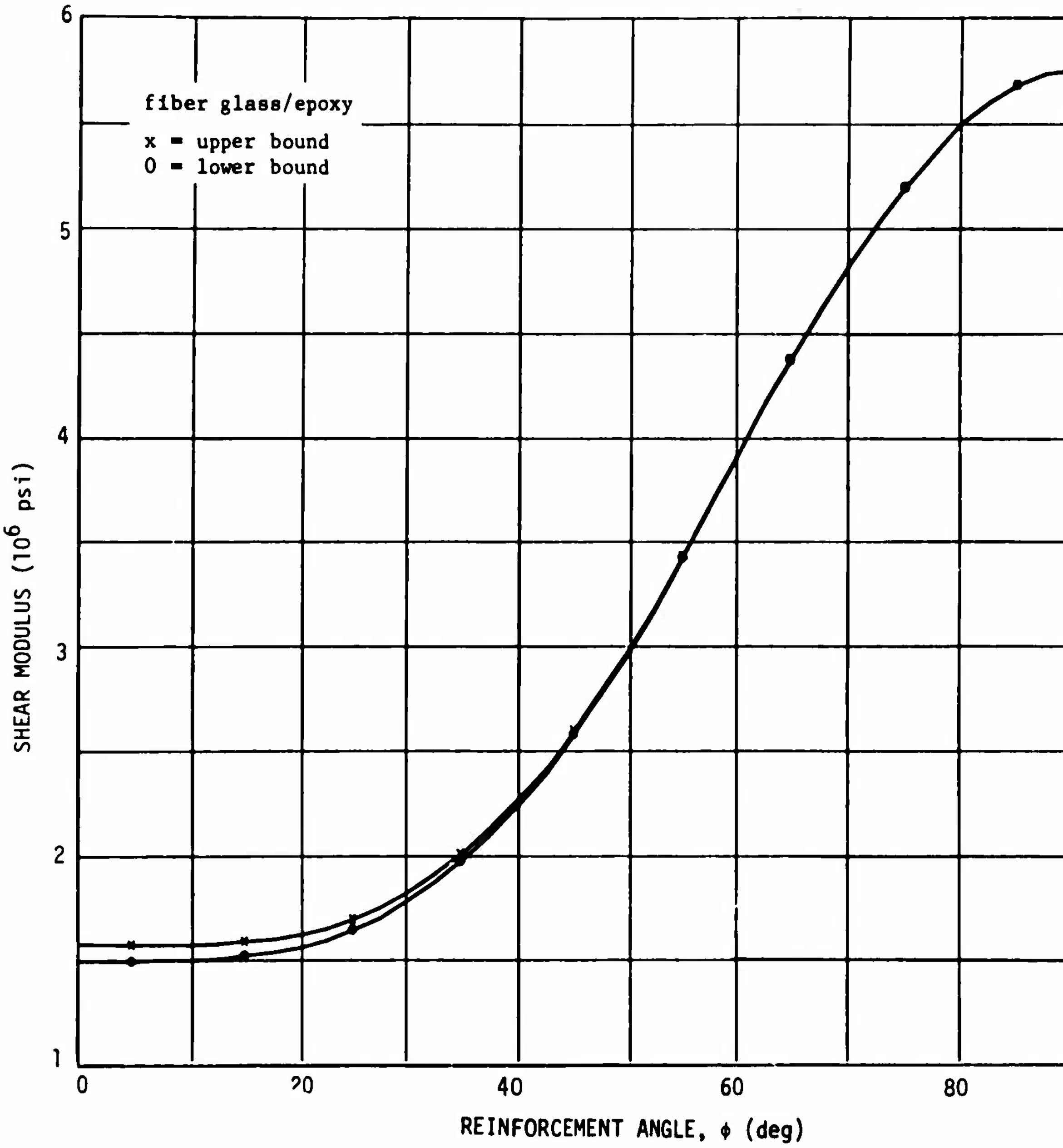

Figure 47. Effective Modulus C_{33}^* vs. Reinforcement Angle
Volume Fraction = 0.5

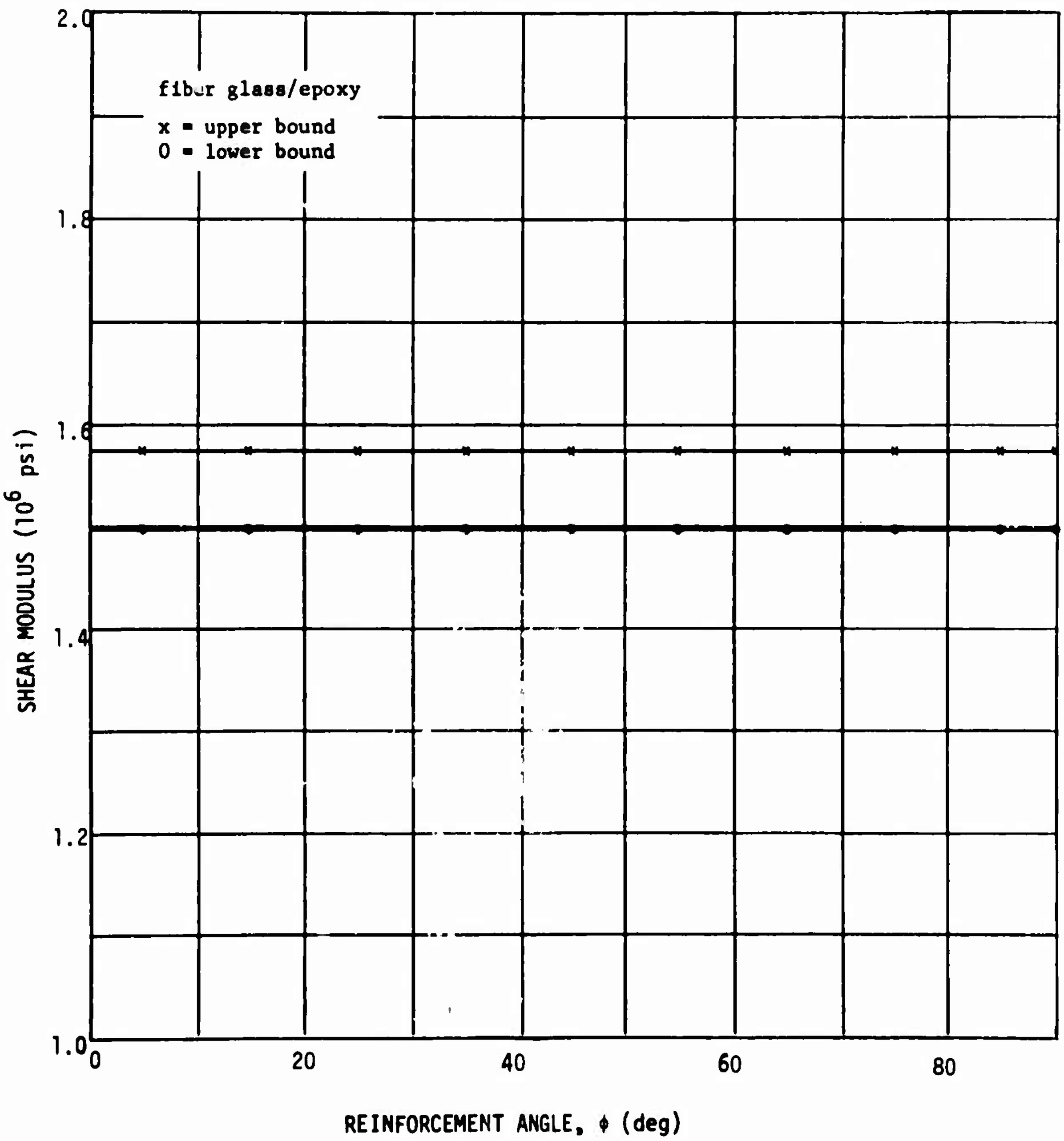

Figure 48. Effective Modulus C^*_{22} vs. Reinforcement Angle
Volume Fraction = 0.5

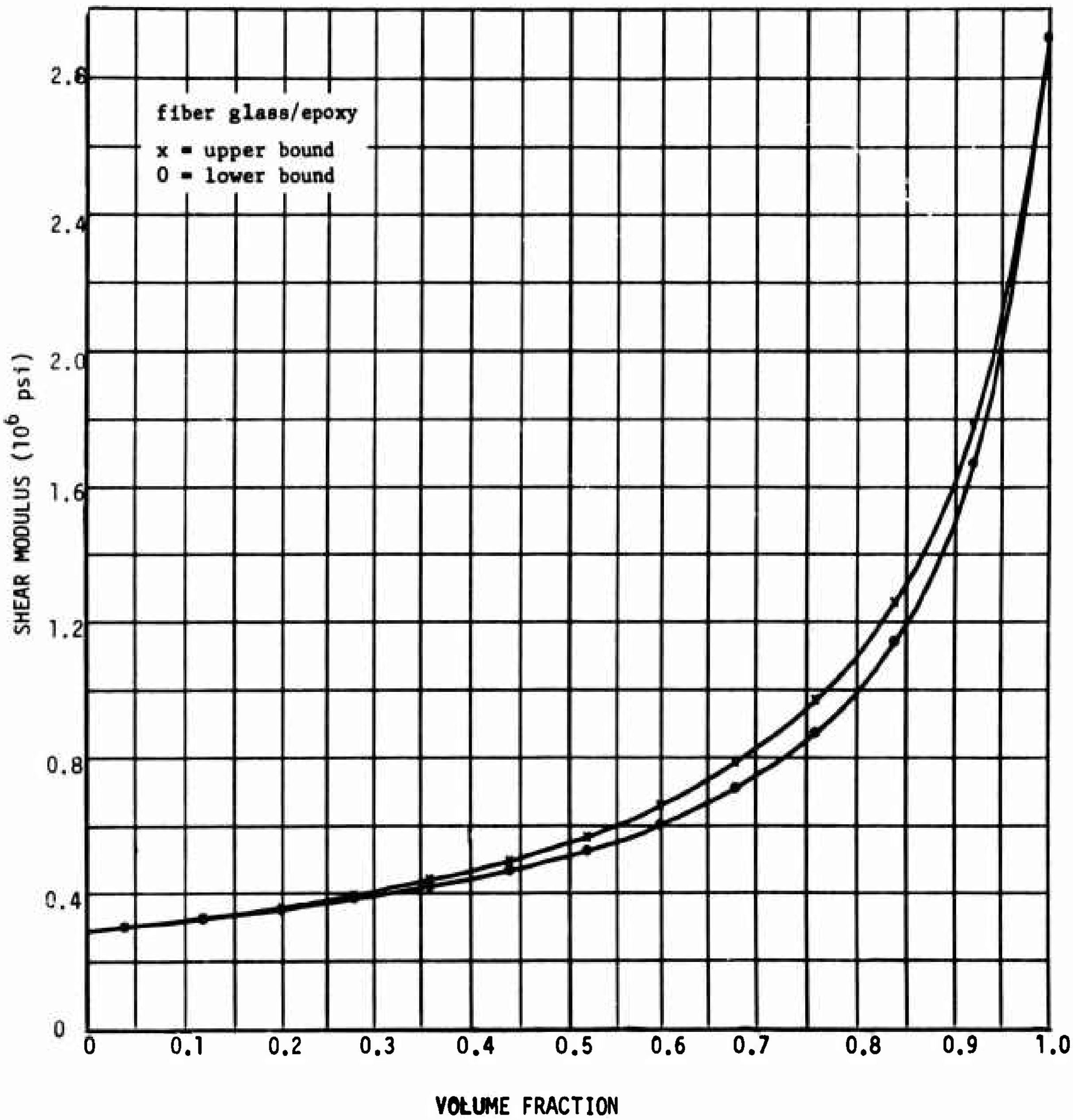

Figure 49. Effective Modulus C_{12}^* vs. Volume Fraction
ϕ = 45 degrees

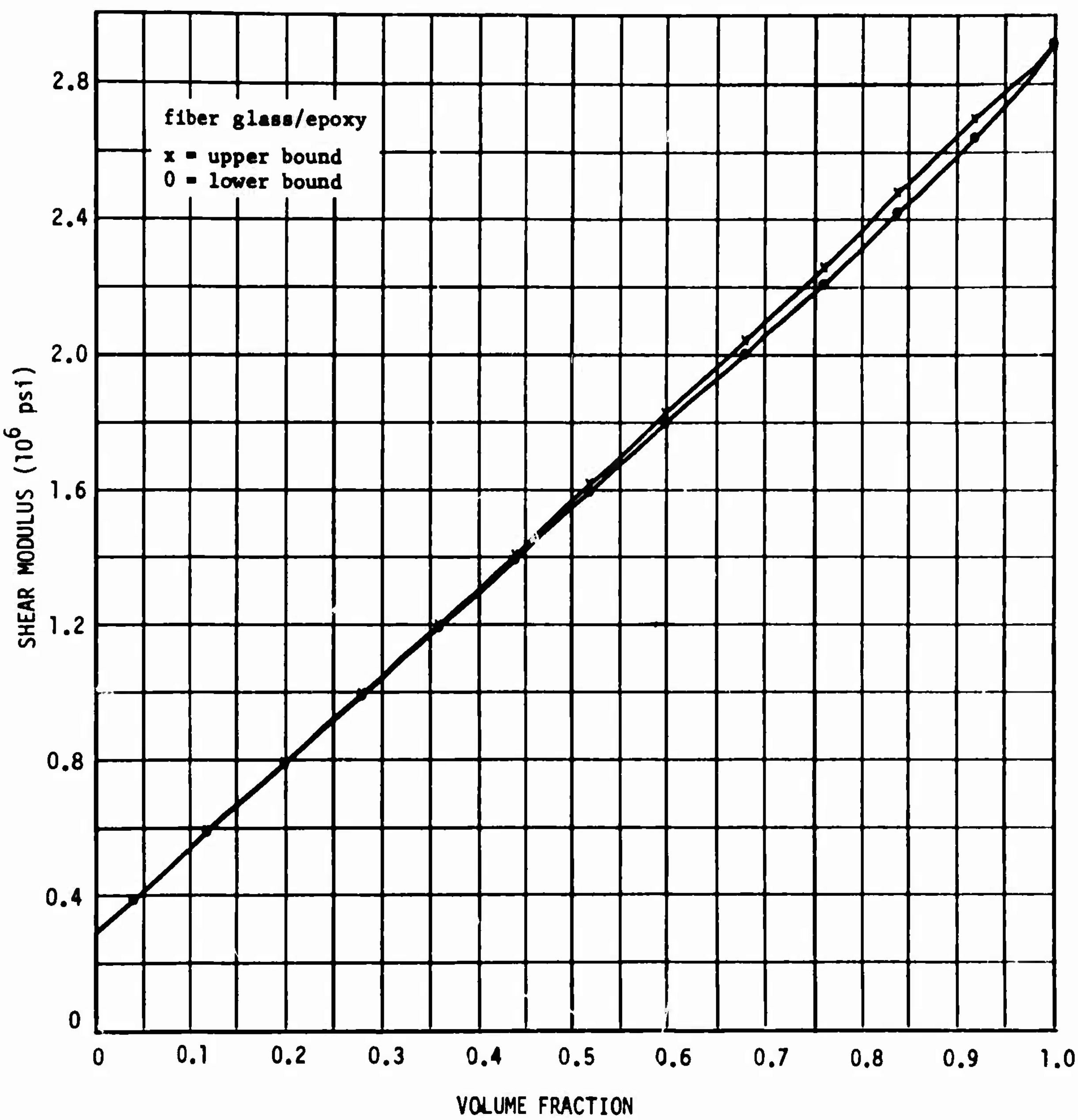

Figure 50. Effective Modulus C_{13}^* vs. Volume Fraction
ϕ = 45 degrees

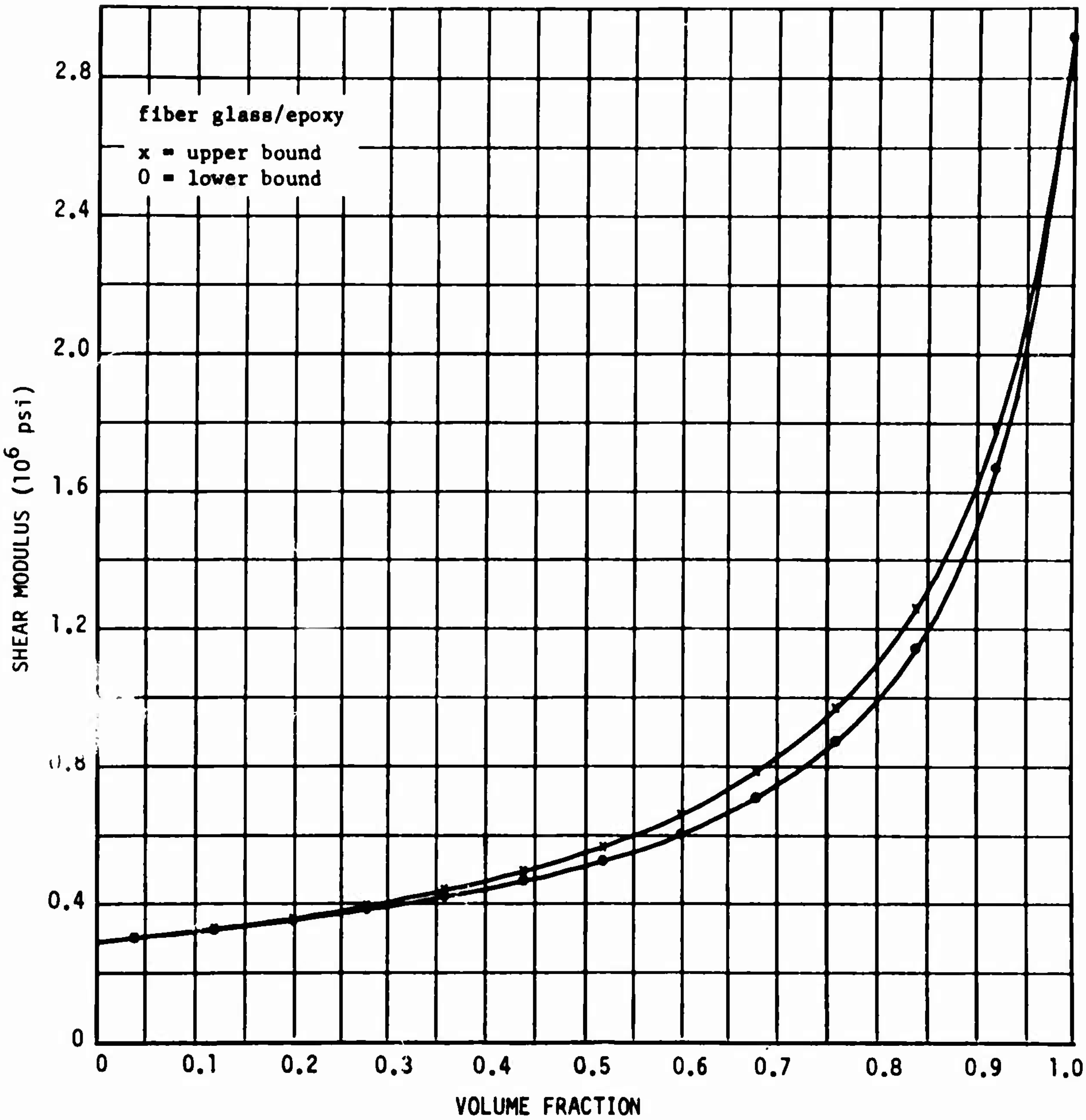

Figure 51. Effective Modulus C^*_{23} vs. Volume Fraction
ϕ = 45 degrees

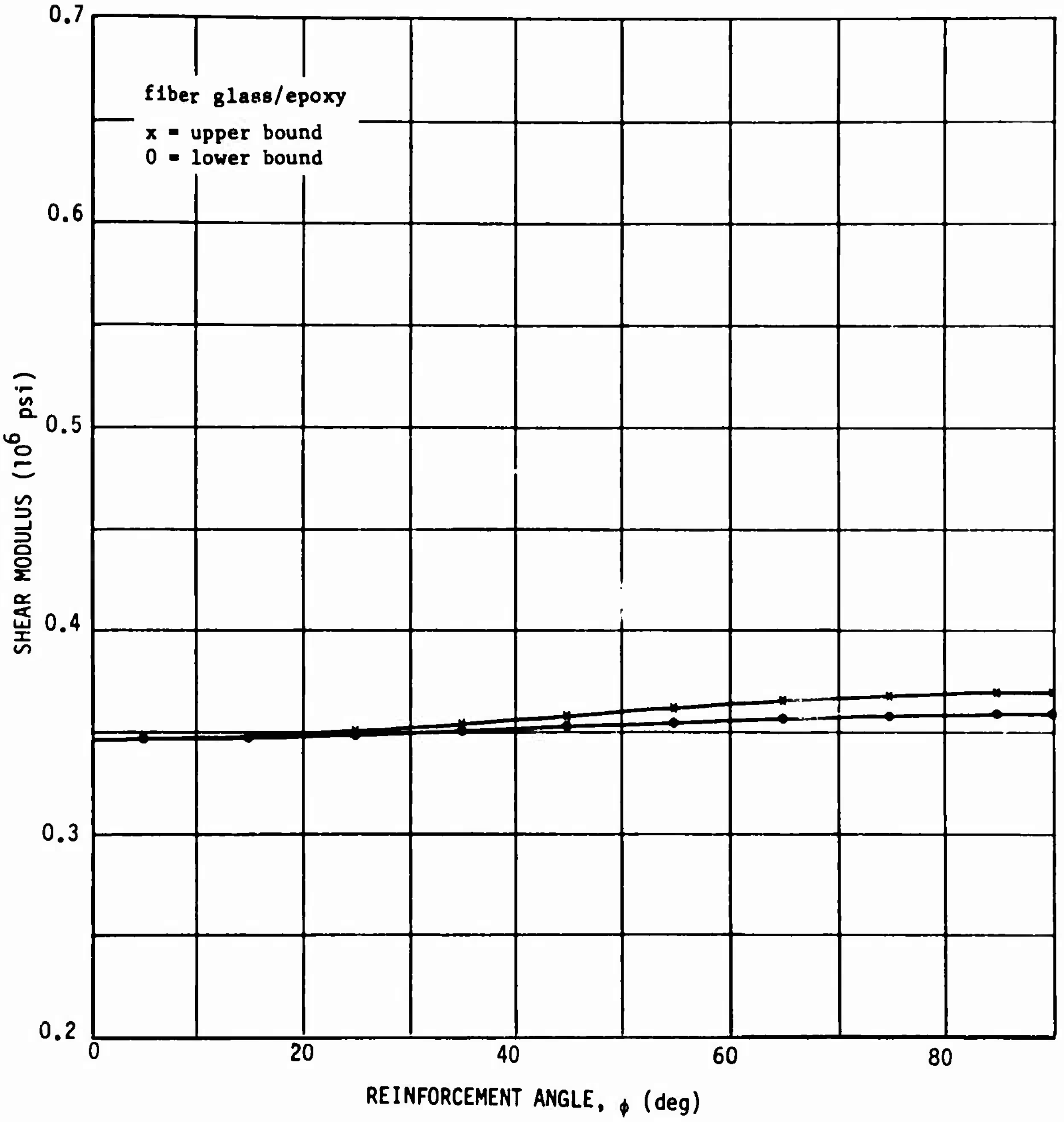

Figure 52. Effective Modulus C_{12}^* vs. Reinforcement Angle
Volume Fraction = 0.2

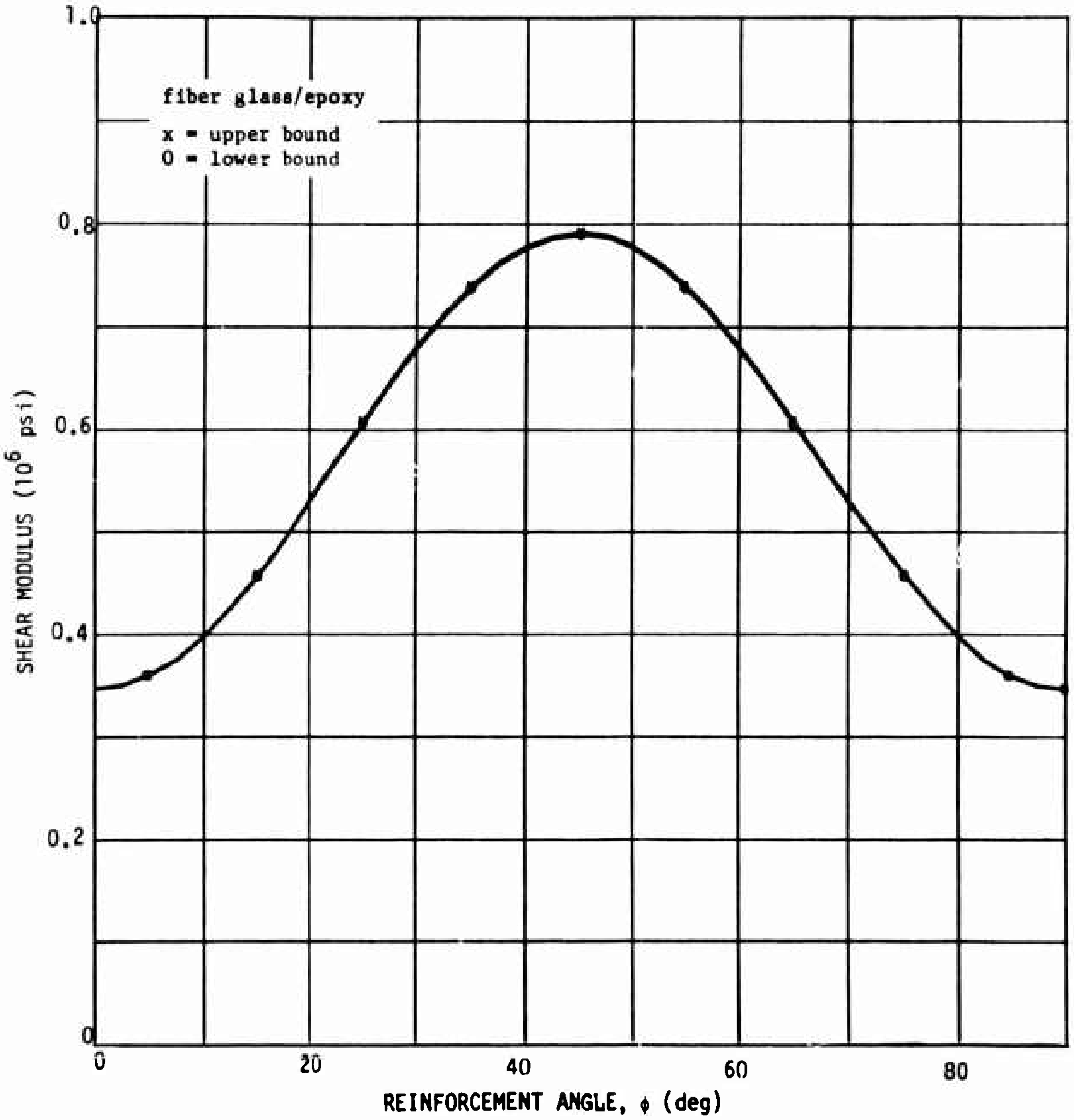

Figure 53. Effective Modulus C_{13}^* vs. Reinforcement Angle
Volume Fraction = 0.2

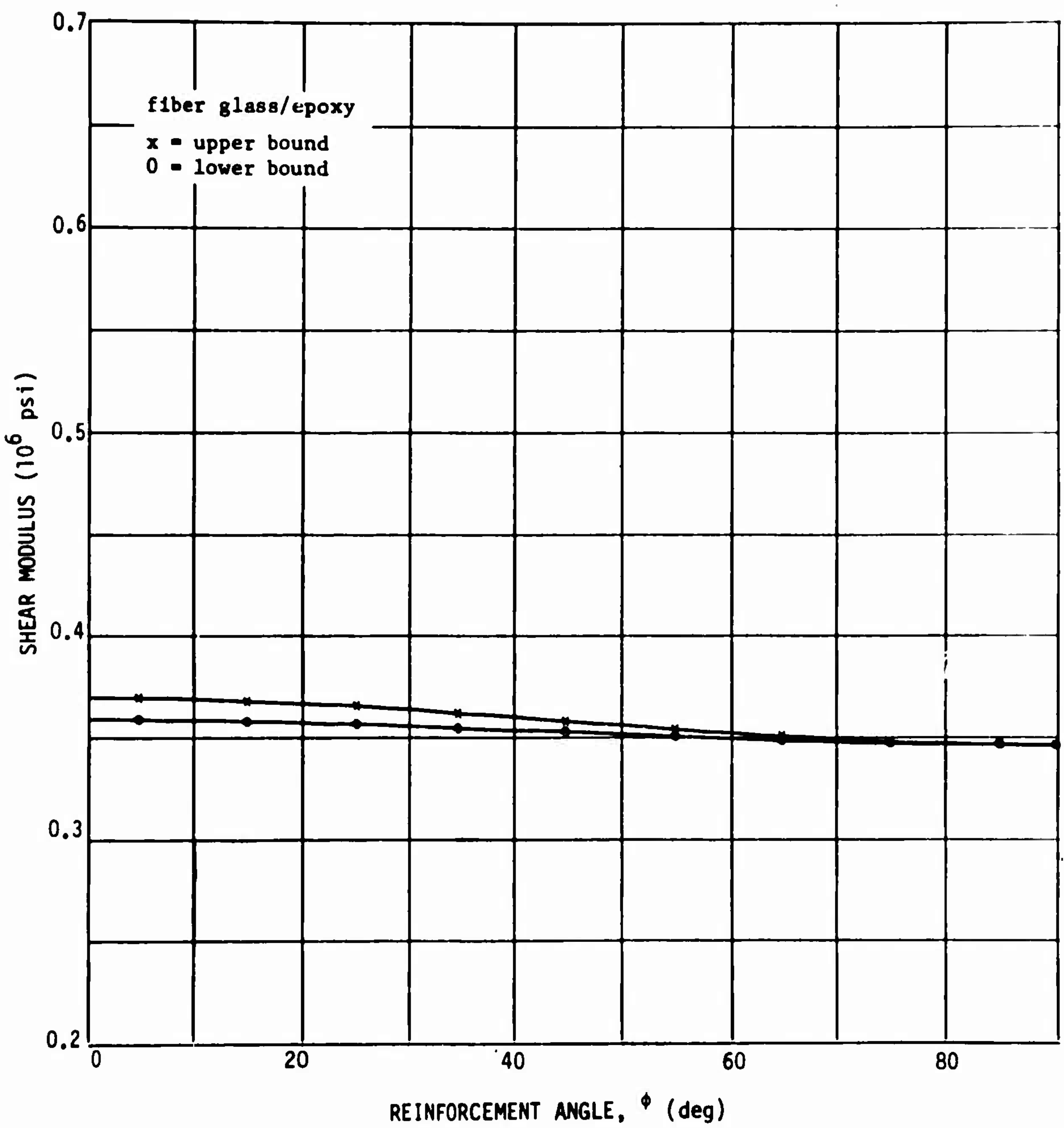

Figure 54. Effective Modulus C_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.2

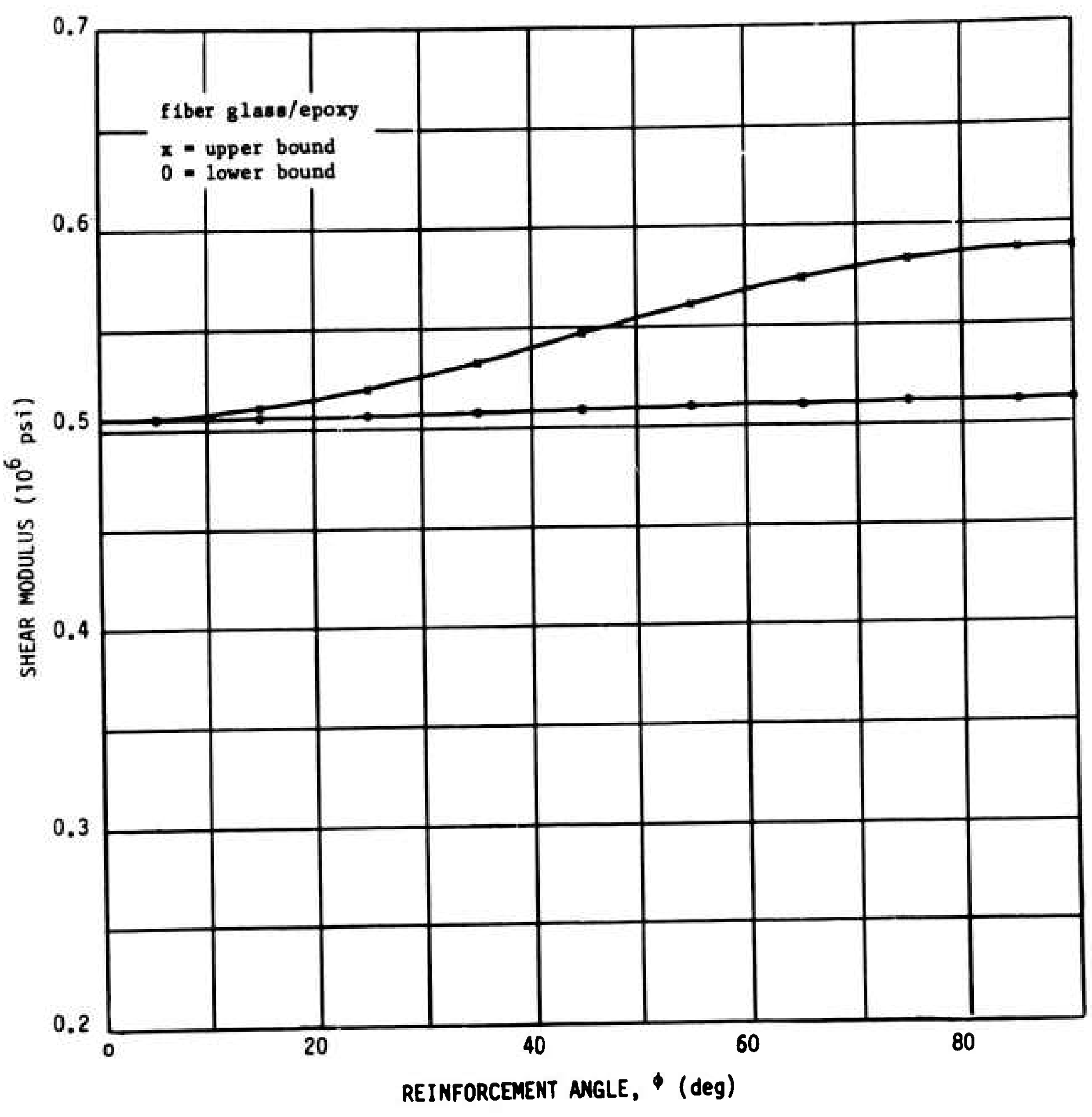

Figure 55. Effective Modulus C_{12}^* vs. Reinforcement Angle
Volume Fraction 0.5

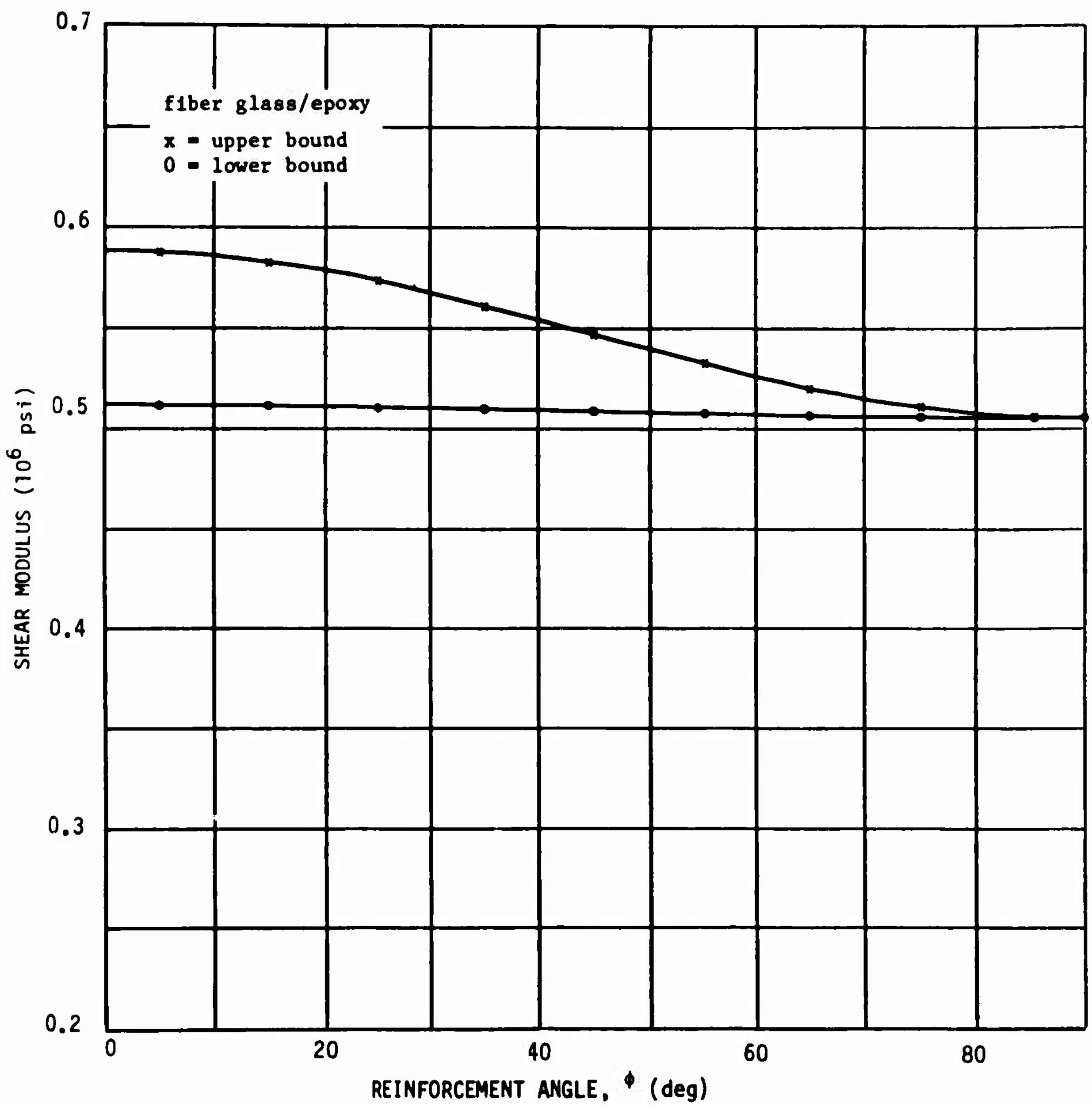

Figure 56. Effective Modulus C_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.5

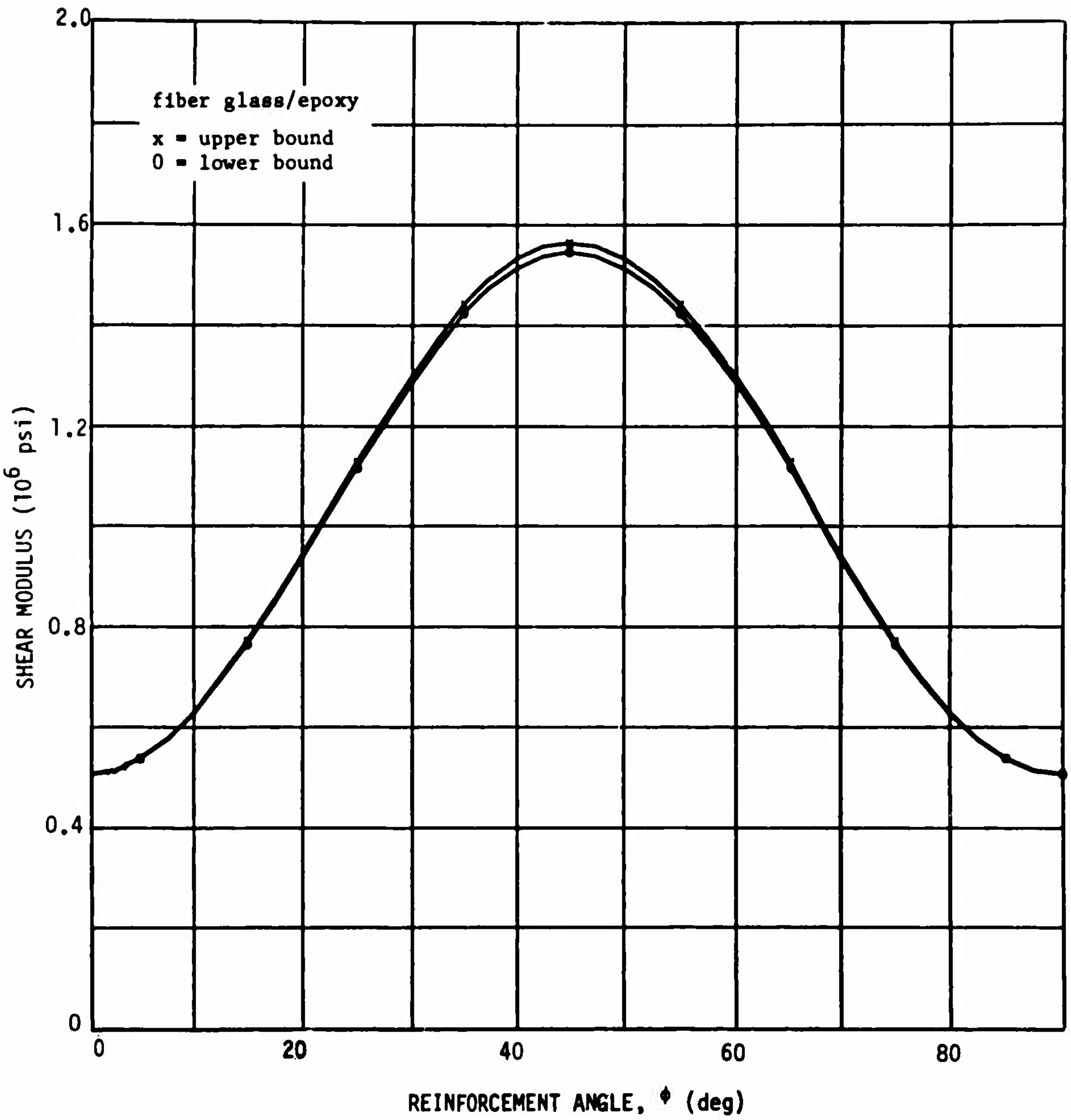

Figure 57. Effective Modulus C_{13}^* vs. Reinforcement Angle
Volume Fraction = 0.5

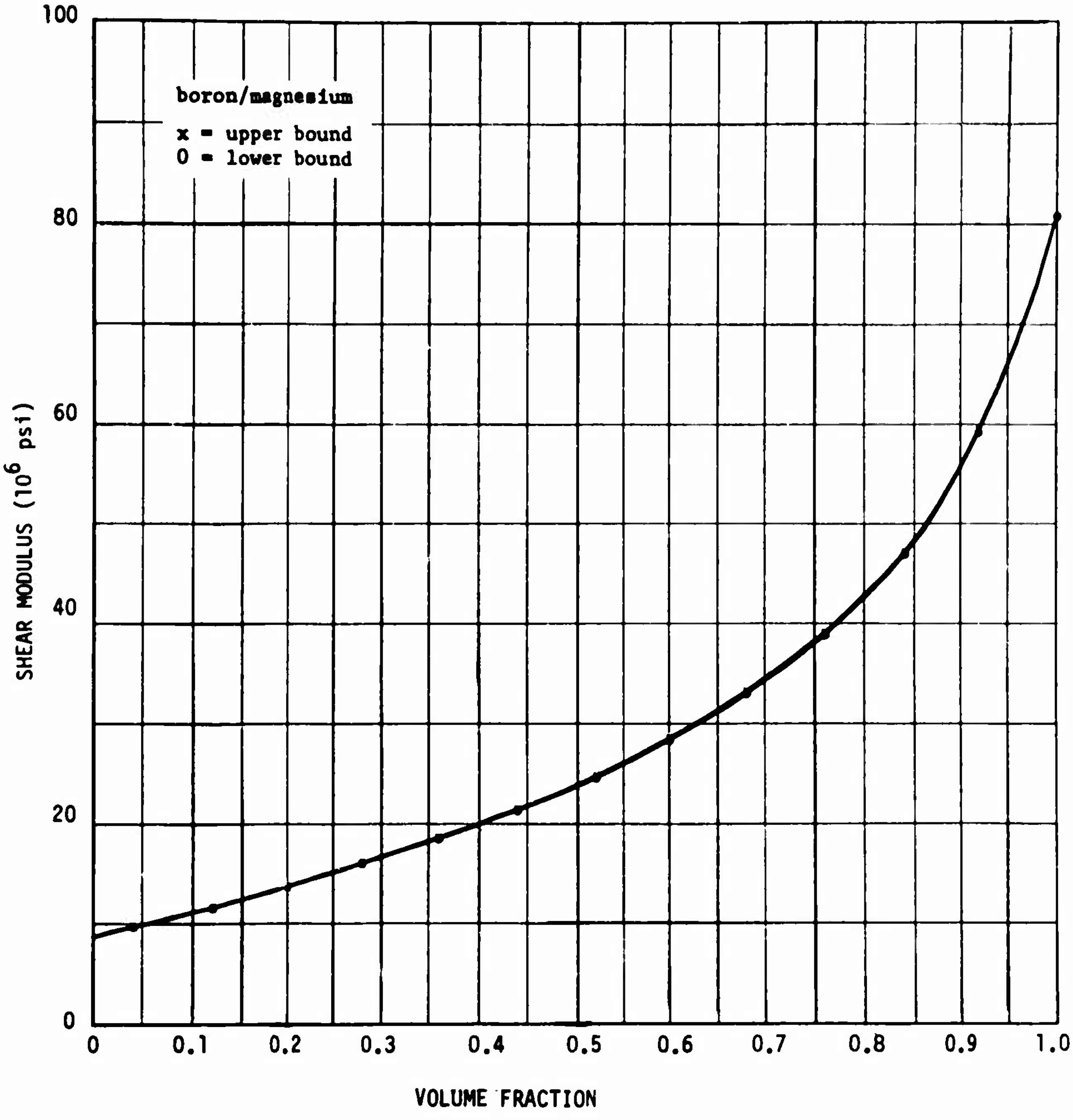

Figure 58. Effective Modulus C^*_{11} vs. Volume Fraction
ϕ = 45 degrees

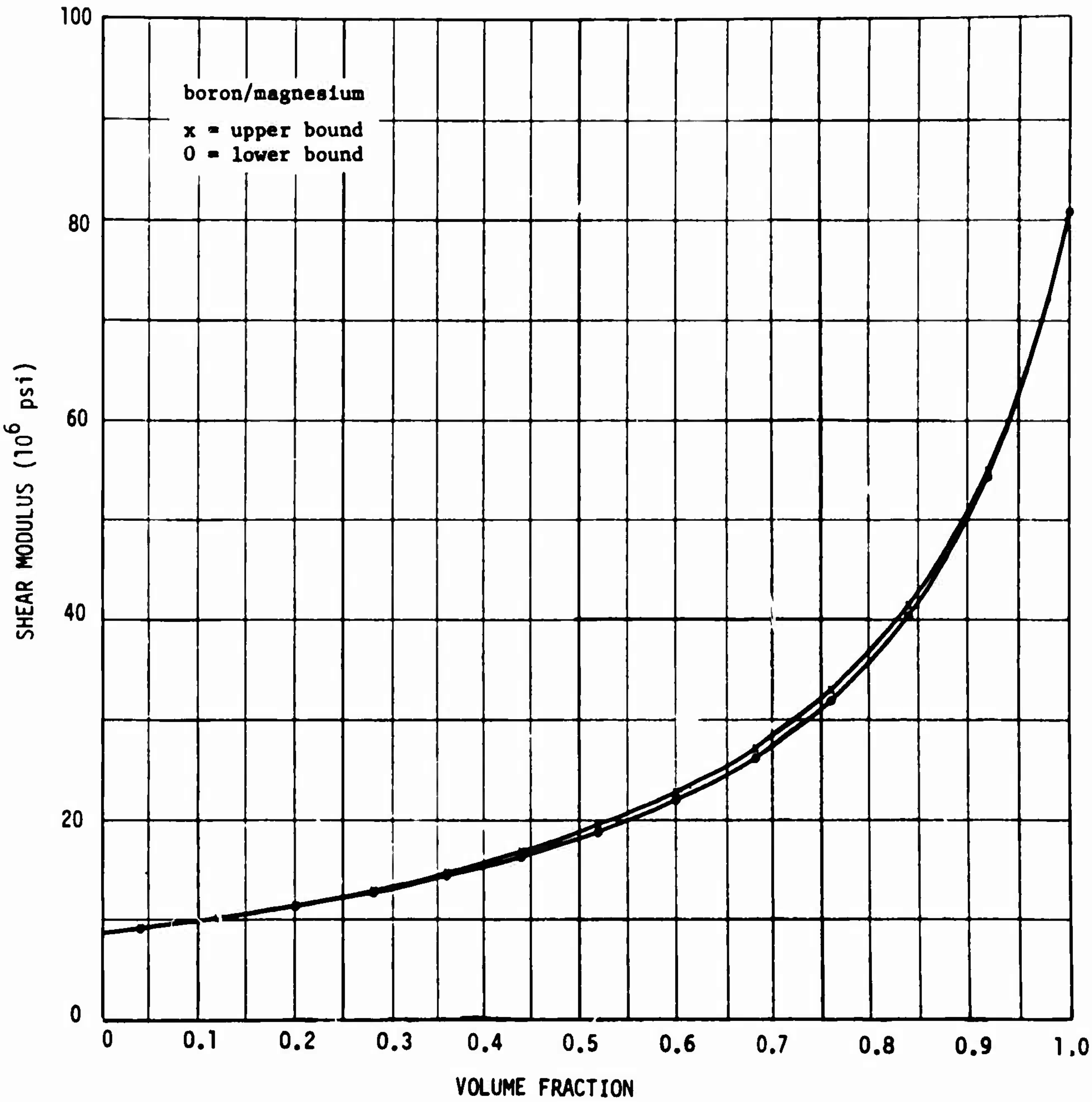

Figure 59. Effective Modulus C_{22}^* vs. Volume Fraction
ϕ = 45 degrees

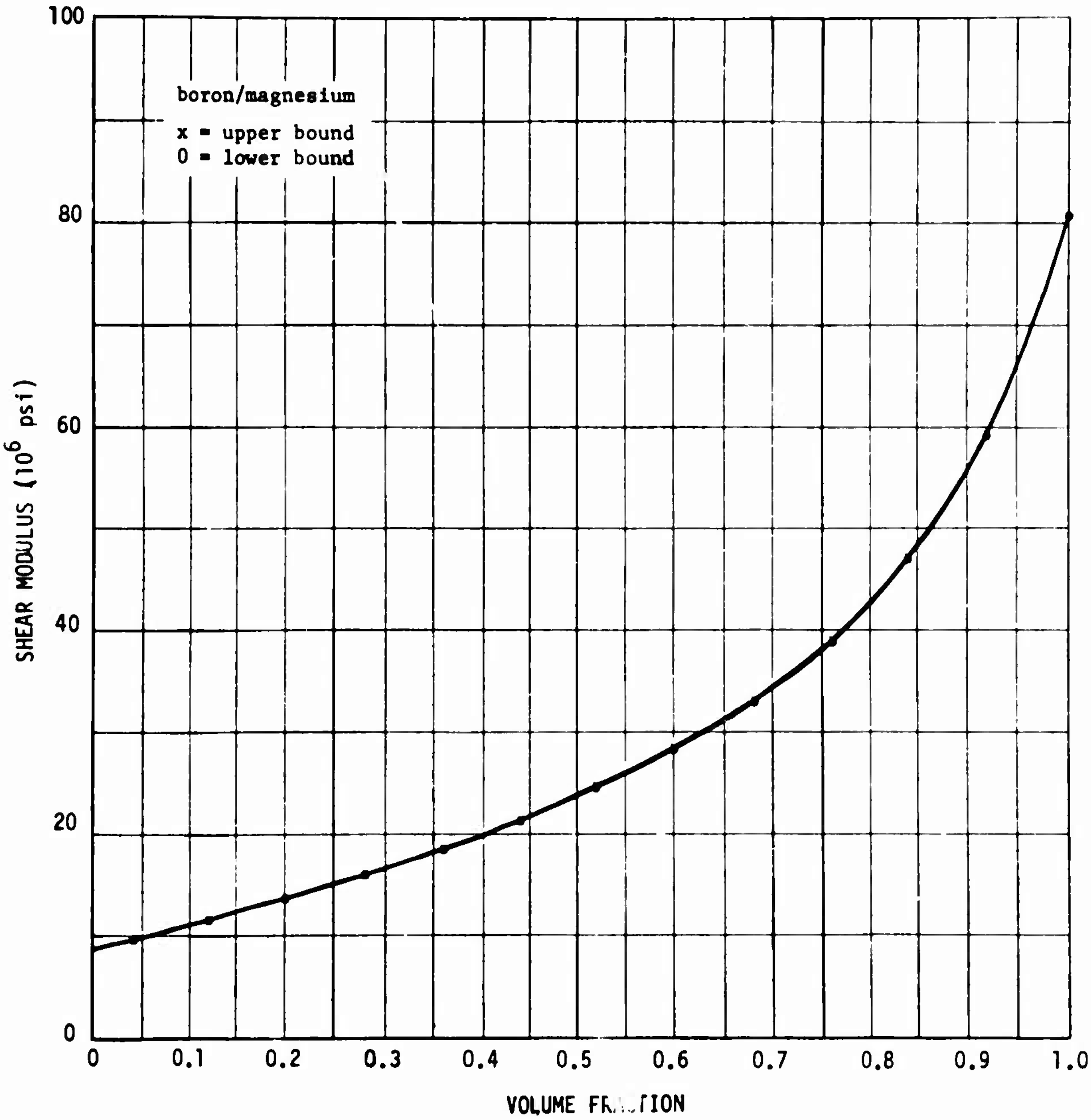

Figure 60. Effective Modulus C_{33}^{*} vs. Volume Fraction
ϕ = 45 degrees

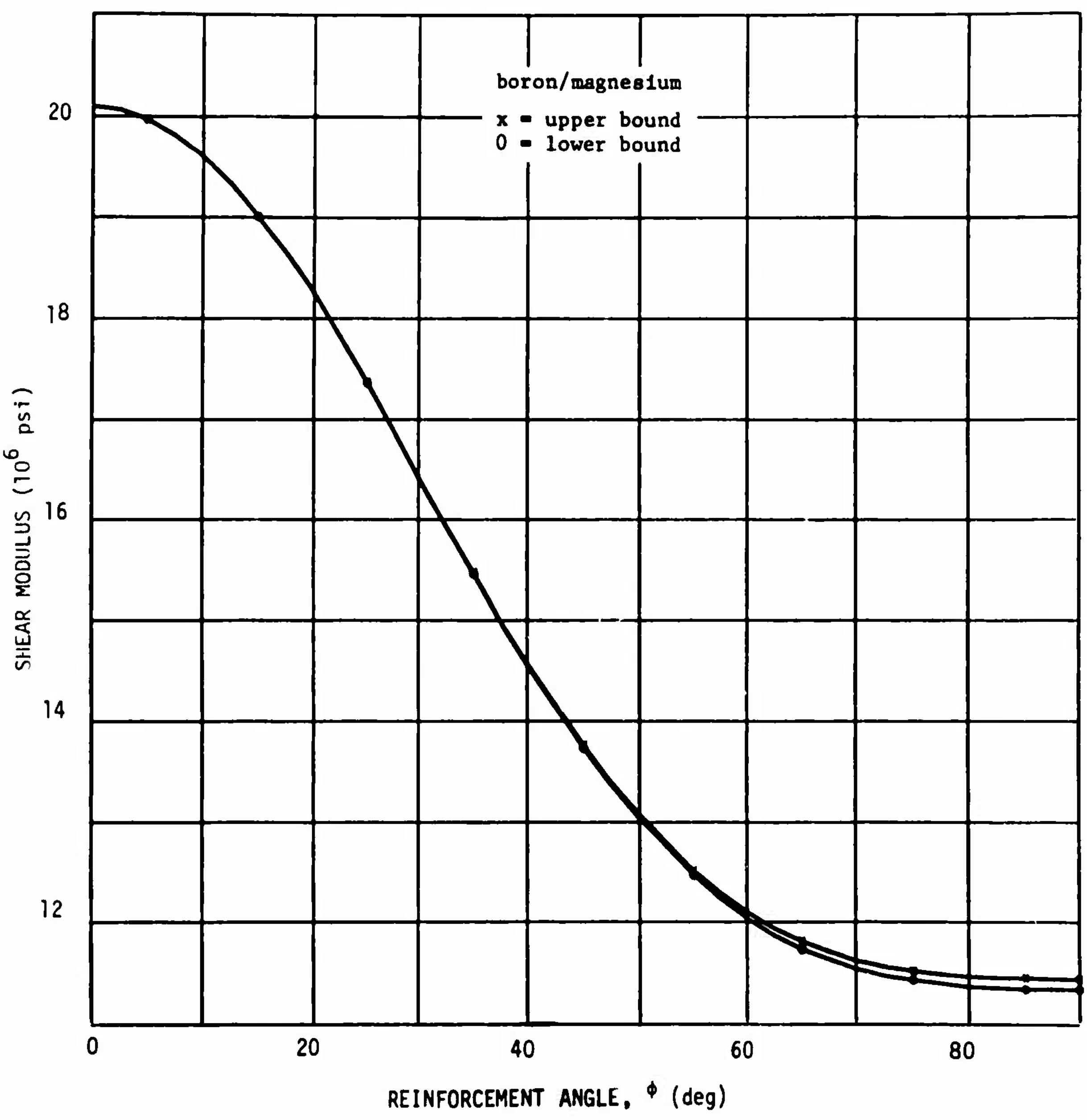

Figure 61. Effective Modulus C_{11}^* vs. Reinforcement Angle
Volume Fraction = 0.2

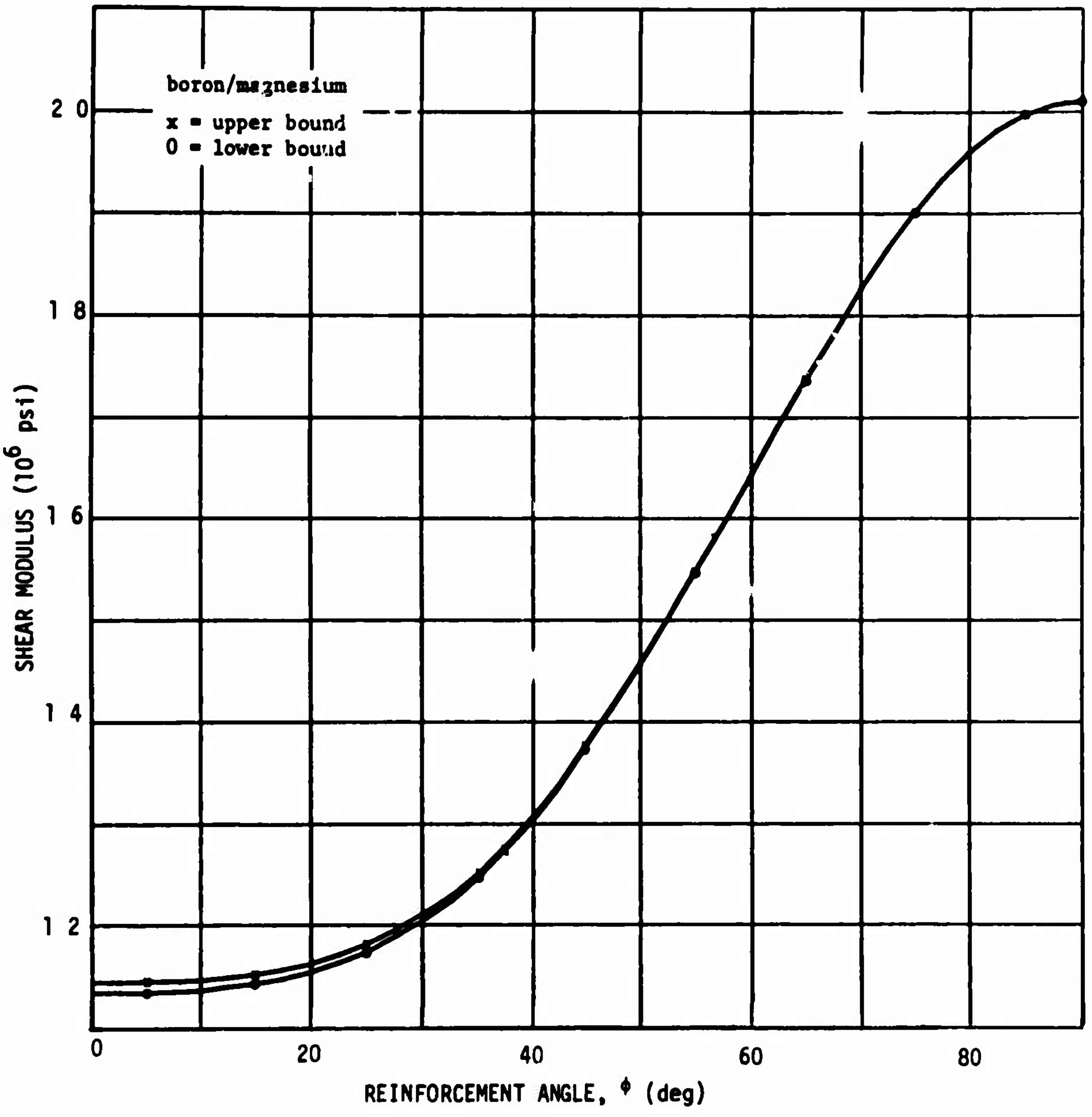

Figure 62. Effective Modulus C_{33}^* vs. Reinforcement Angle
Volume Fraction = 0.2

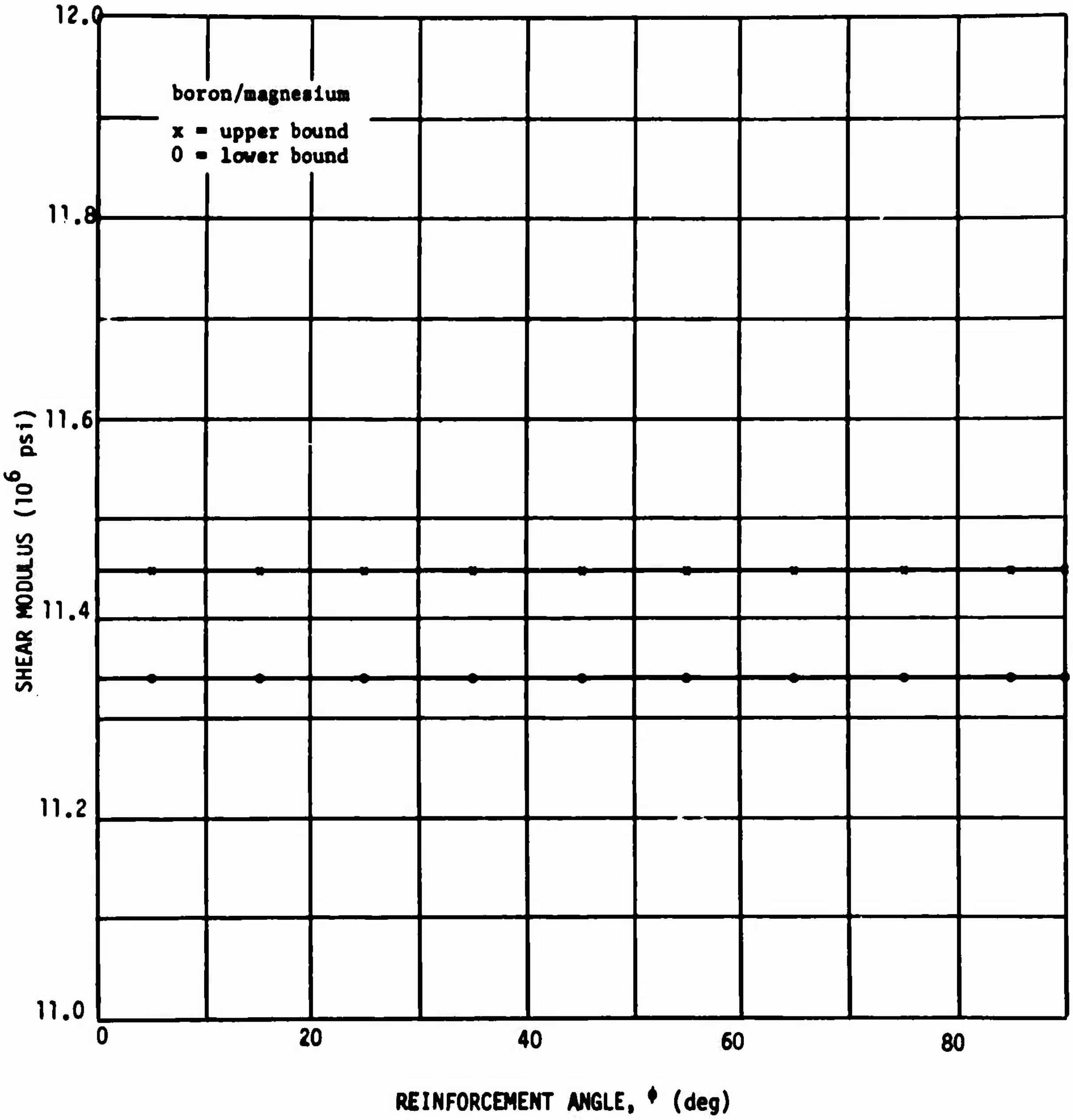

Figure 63. Effective Modulus C_{22}^* vs. Reinforcement Angle
Volume Fraction = 0.2

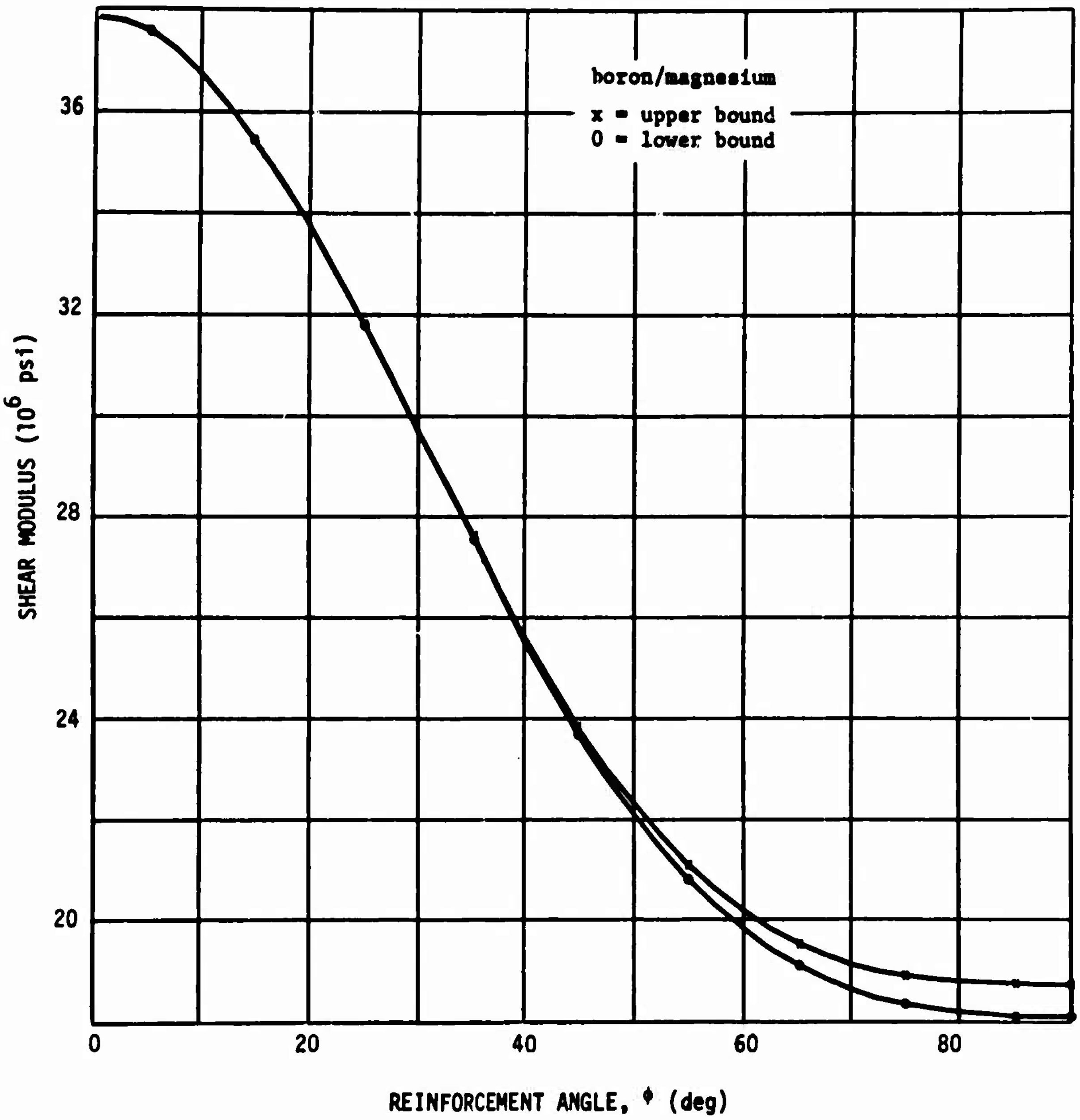

Figure 64. Effective Modulus C_{11}^* vs. Reinforcement Angle
Volume Fraction = 0.5

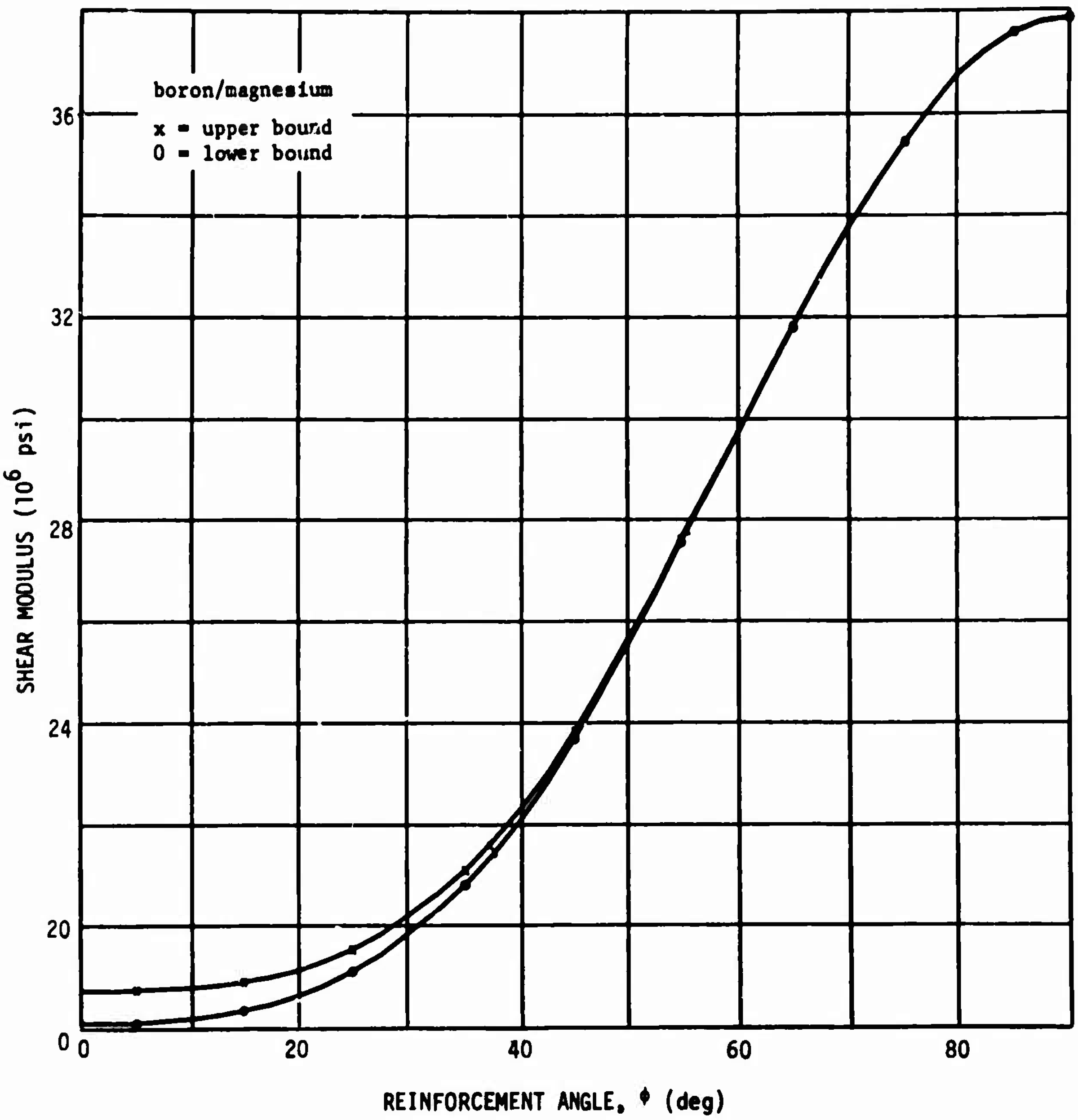

Figure 65. Effective Modulus C_{33}^{*} vs. Reinforcement Angle
Volume Fraction = 0.5

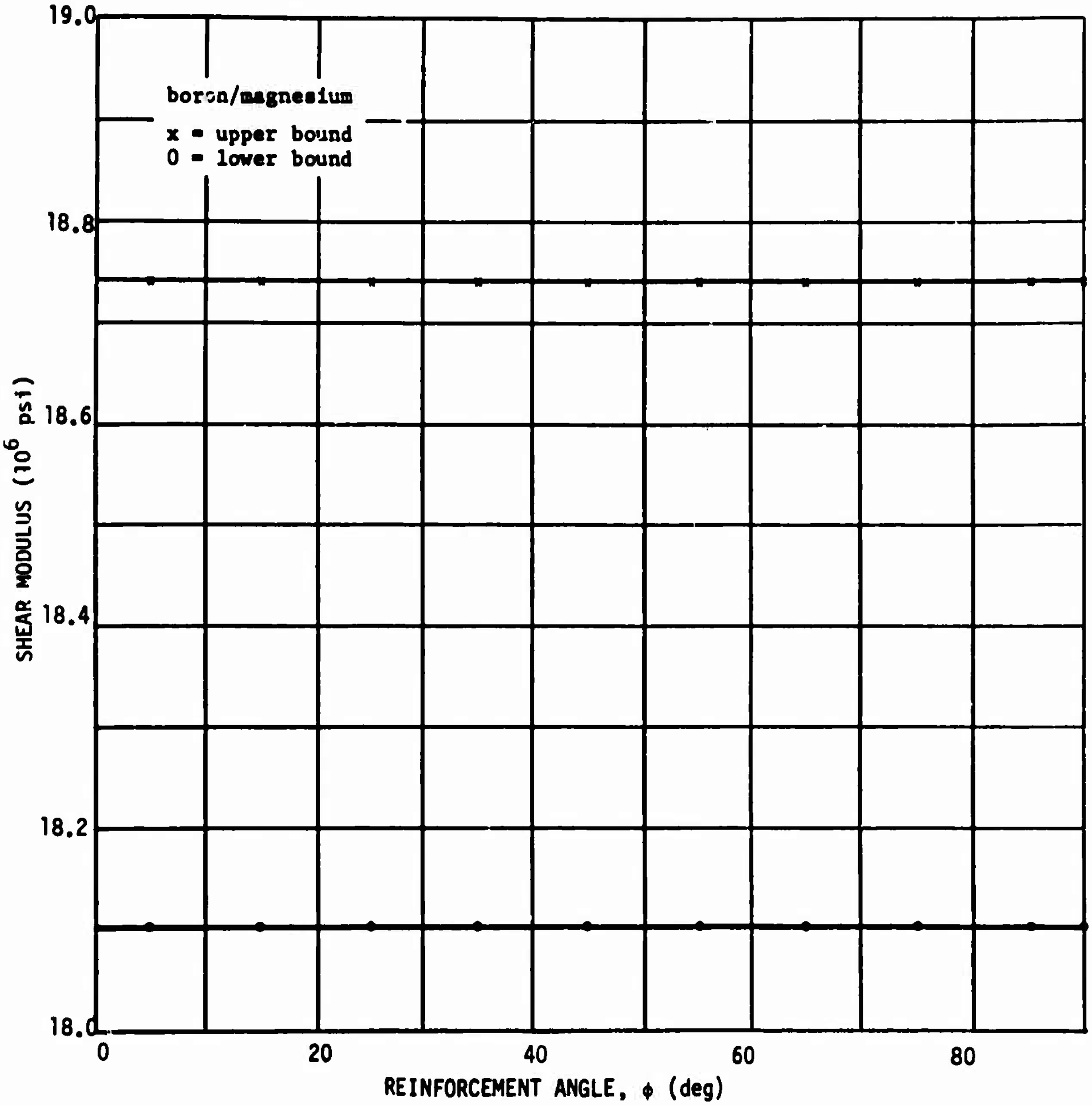

Figure 66. Effective Modulus C^*_{22} vs. Reinforcement Angle
Volume Fraction = 0.5

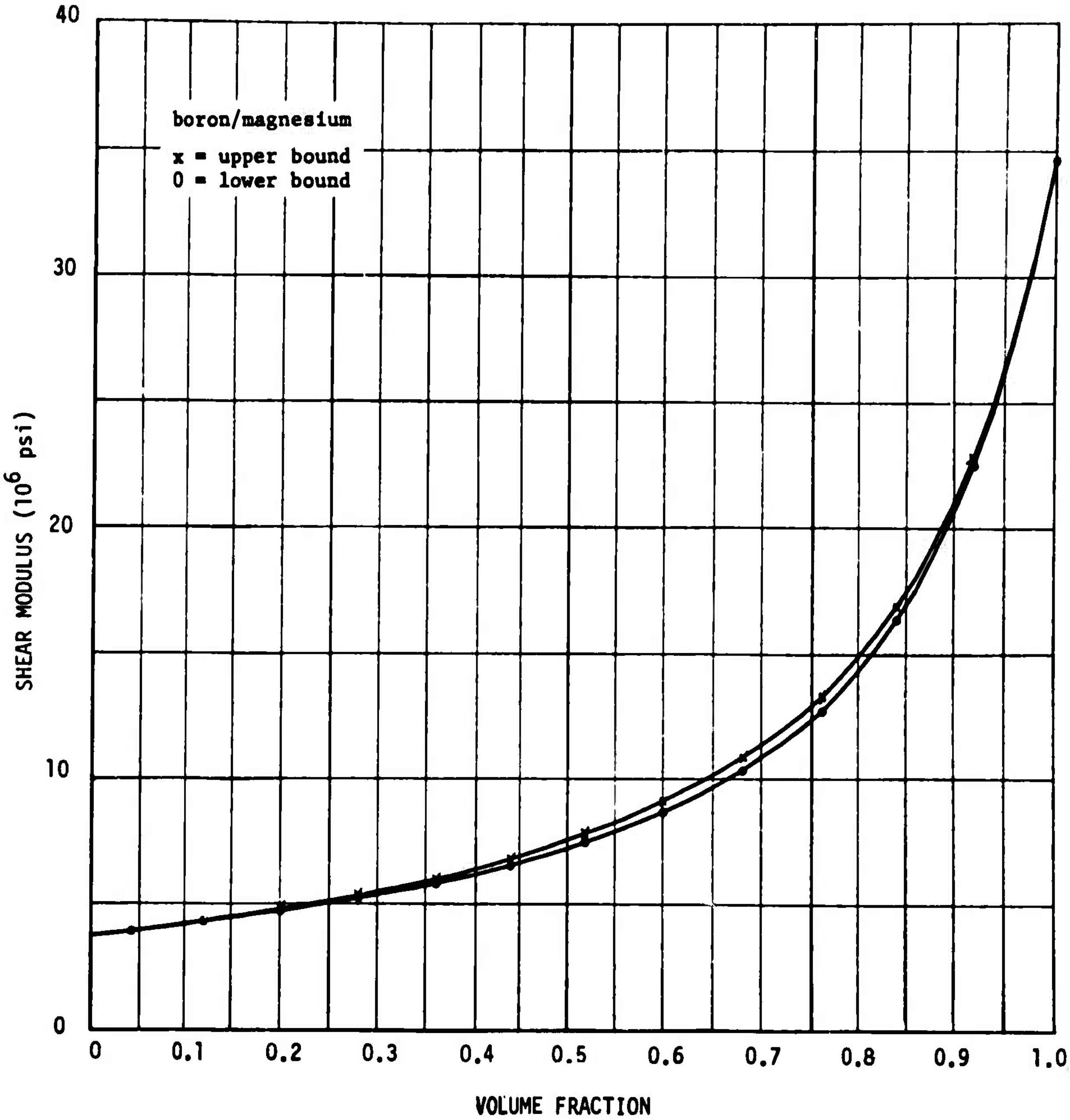

Figure 67. Effective Modulus C_{12}^* vs. Volume Fraction
ϕ = 45 degrees

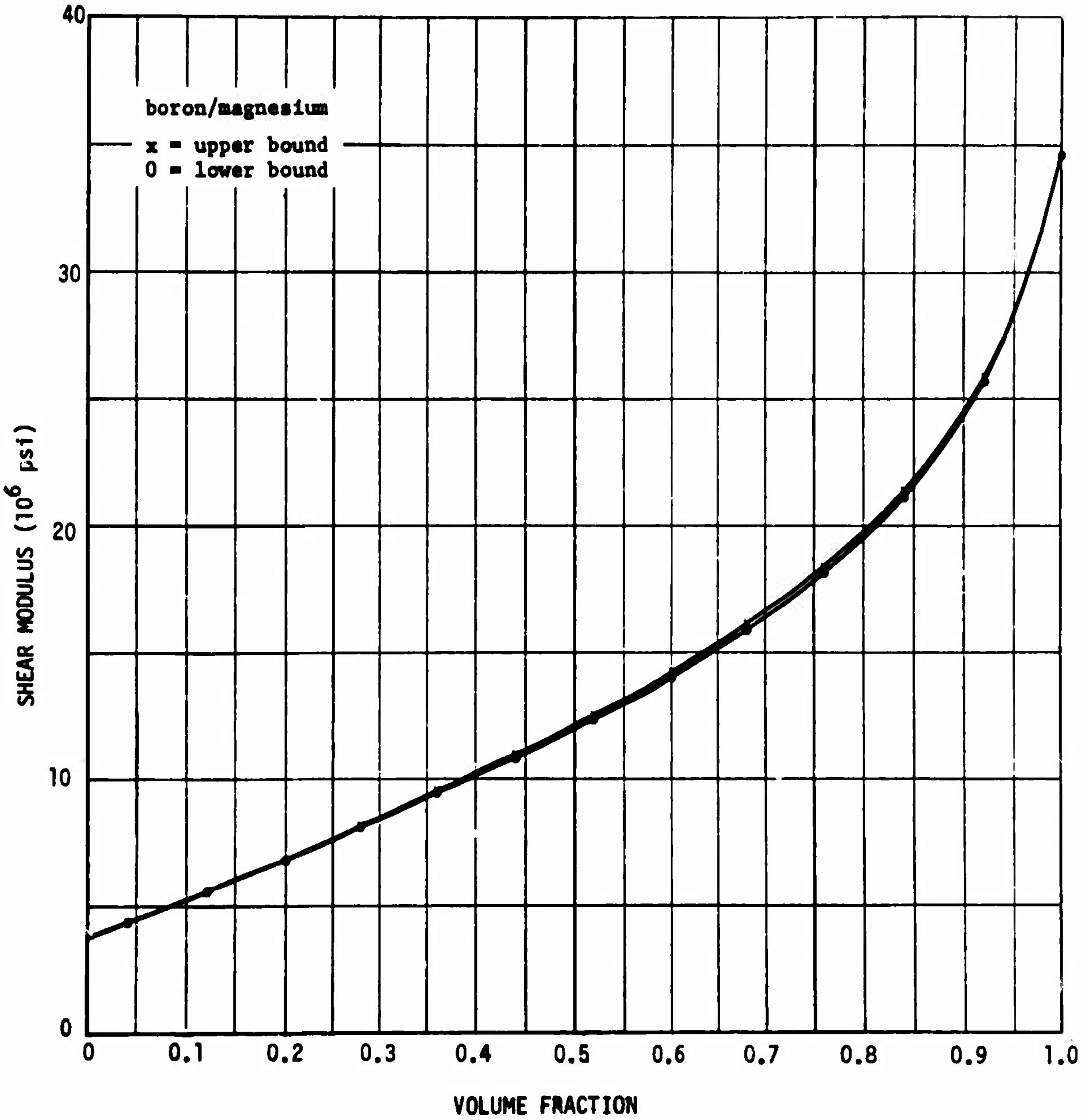

Figure 68. Effective Modulus C^*_{13} vs. Volume Fraction
ϕ = 45 degrees

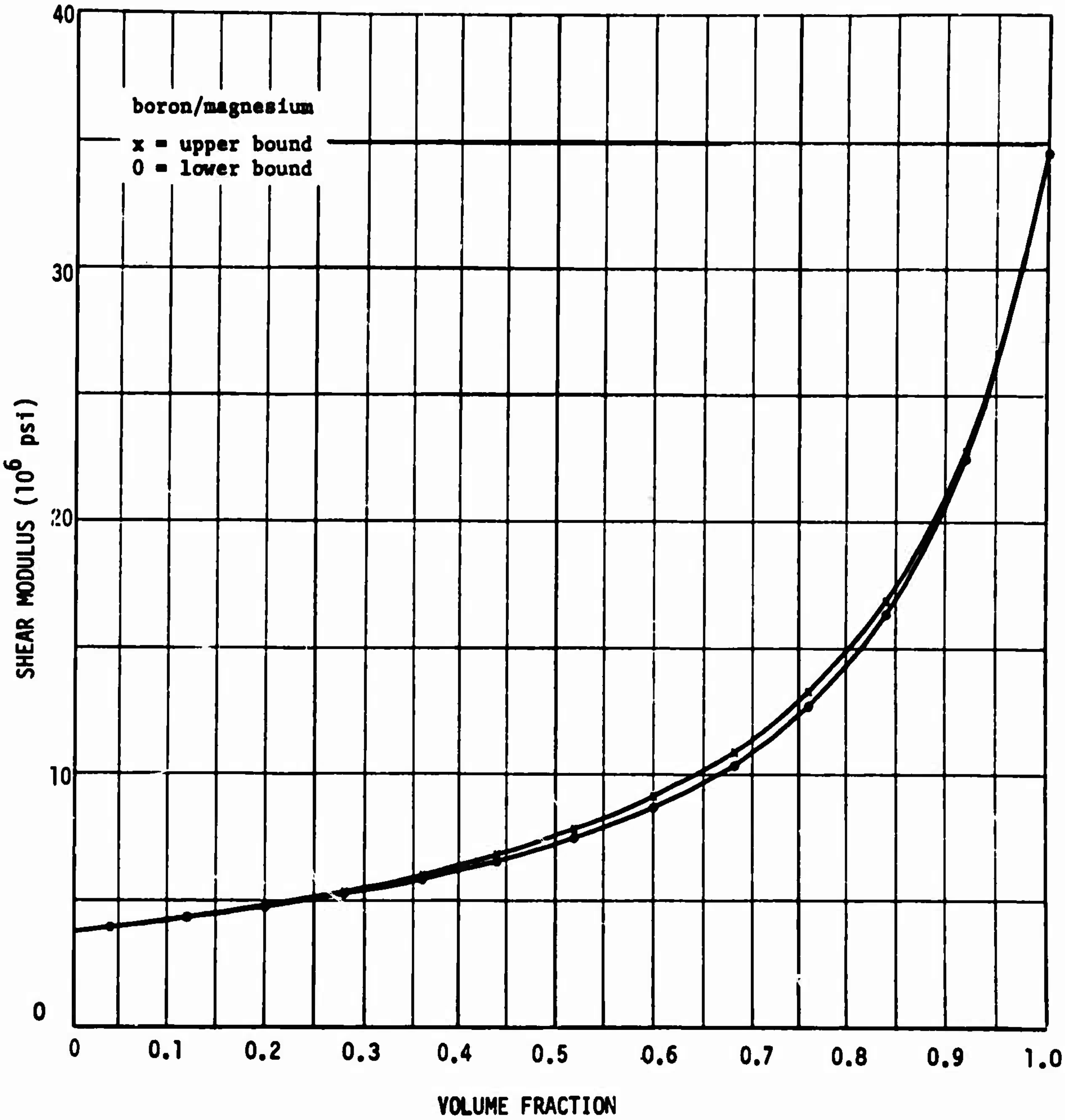

Figure 69. Effective Modulus C_{23}^* vs. Volume Fraction
ϕ = 45 degrees

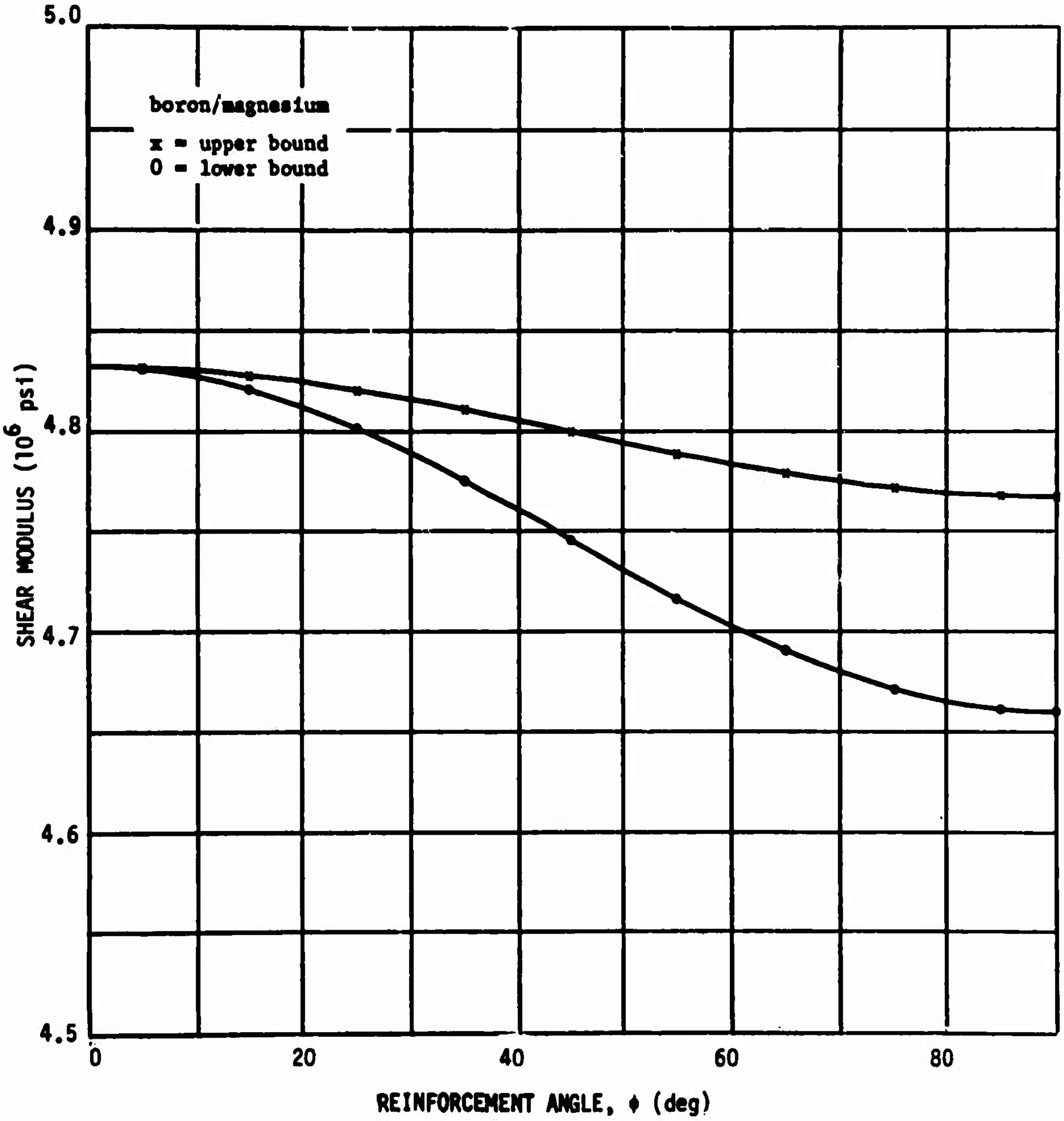

Figure 70. Effective Modulus C_{12}^* vs. Reinforcement Angle
Volume Fraction = 0.2

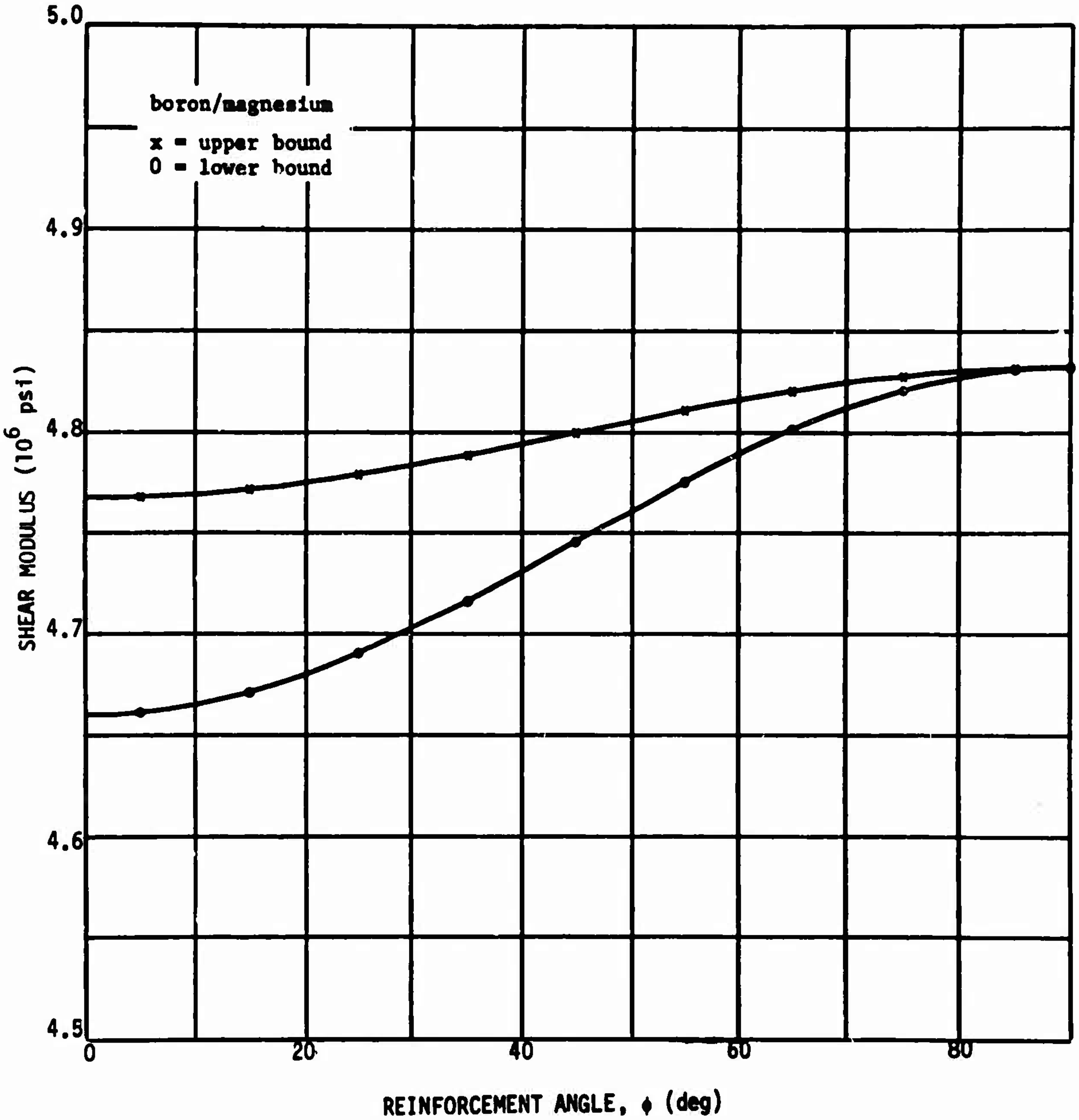

Figure 71. Effective Modulus C_{23}^* vs. Reinforcement Angle
Volume Fraction = 0.2

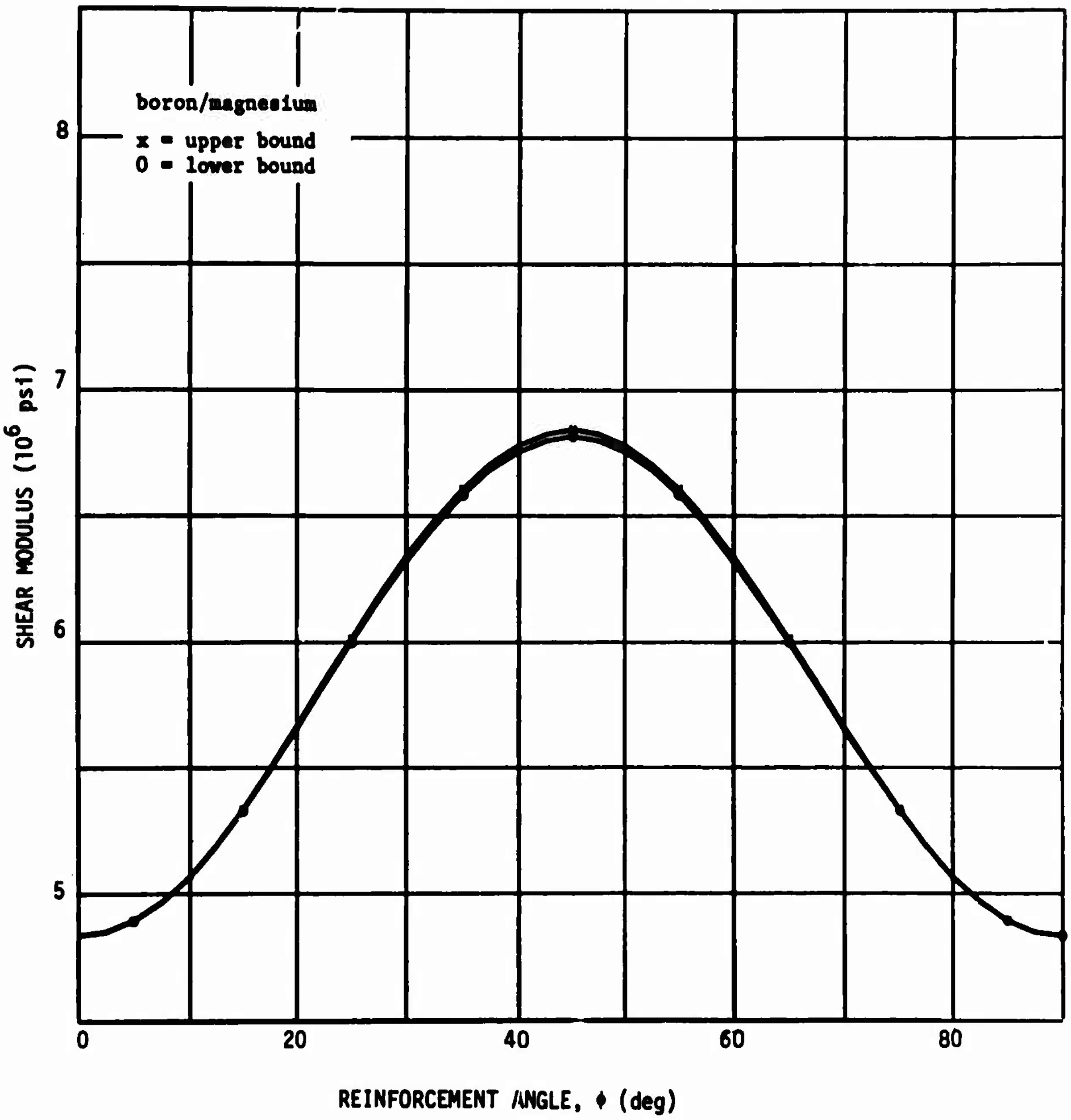

Figure 72. Effective Modulus C^*_{13} vs. Reinforcement Angle
Volume Fraction = 0.2

12. BOUNDS FOR NORMAL EFFECTIVE ELASTIC COMPLIANCES FOR COMPOSITE CYLINDER MODEL

The problem now arises: Can bounds on the normal effective compliances be generated on the basis of the known bounds on the effective elastic moduli?

Analytical treatment of this problem is unfortunately prohibitively difficult. The analytical procedure would be to express the compliance S^*_{ij} by inversion of the matrix C^*_{ij} and then to investigate whether the resulting expressions are monotonically increasing or decreasing in terms of C^*_{ij} variations. The complexity of the expressions for S^*_{ij} in terms of C^*_{ij} rules out such an investigation.

The procedure which is here adopted is a numerical one. The compliances C^*_{ij} as given by (168) through (174) are computed for a boron/epoxy composite. The elastic properties of matrix and fibers are given by

$E_m = 0.5 \times 10^6$ psia $\qquad E_f = 57.5 \times 10^6$ psi

$\nu_m = 0.35 \qquad \nu_f = 0.2$

The calculations will be performed for four values of G_t. These are the lower and upper bounds on G_t(i.e., $G_t^{(-)}$ and $G_t^{(+)}$), as given by uniaxially composite theory, as described above, and by two intermediate values

$$G_t^{(1)} = G_t^{(-)} + \frac{1}{3}\left(G_t^{(+)} - G_t^{(-)}\right) , \tag{181}$$

$$G_t^{(2)} = G_t^{(-)} + \frac{2}{3}\left(G_t^{(+)} - G_t^{(-)}\right) . \tag{182}$$

As has been pointed out in Section 11, (168), (170), (171), and (174) are monotonically increasing functions of G_t, while (169) and (173) are monotonically decreasing functions of G_t. Thus, for the values $G_t^{(-)}$ and $G_t^{(+)}$, the bounds for the C^*_{ij} are obtained, while the values (181) and (182) give intermediate values. C^*_{ij} results for $G_t^{(-)}$ are shown in Table I; for $G_t^{(1)}$, in Table III; for $G_t^{(2)}$, in Table IV; and for $G_t^{(+)}$, in Table II.

Now the C^*_{ij} matrix is numerically inverted in each case, whereby four sets of S^*_{ij} values are obtained. These results are shown in Tables V through VIII.

According to the numerical results, the S_{ij}^* are monotonic functions of G_t. It is seen that lower/upper bounds for C_{ij}^* generate upper/lower bounds for the diagonal compliances S_{11}^*, S_{22}^*, and S_{33}^*. Because of (15) upper/lower bounds on S_{11}^*, S_{22}^*, and S_{33}^* correspond to lower/upper bounds on E_1^*, E_2^*, and E_3^*. Thus, lower/upper bounds for C_{ij}^* generate lower/upper bounds for E_1^*, E_2^*, and E_3^*.

For the tangential compliances S_{12}^*, S_{23}^*, and S_{31}^*, there does not seem to be a similar rule. C_{ij}^* bounds do generate $S_{ij\,(i\neq j)}^*$ bounds, but it is not known beforehand whether lower/upper C_{ij}^* bounds will generate lower/upper S_{ij}^* bounds, or vice versa.

It is worthwhile to show how Poisson's ratio bounds are obtained. Assume, for example, that S_{11}^* and S_{12}^* bounds are known by numerical calculation. Thus,

$$S_{11}^{*(-)} \leq S_{11}^* \leq S_{11}^{*(+)} \quad , \tag{183}$$

$$S_{12}^{*(-)} \leq S_{12}^* \leq S_{12}^{*(+)} \quad . \tag{184}$$

The form (15 a)

$$E_1^{*(-)} = \frac{1}{S_{11}^{*(+)}} \leq E_1^* \leq \frac{1}{S_{11}^{*(-)}} = E_1^{*(+)} \quad , \tag{185}$$

which yields the E_1^* bounds. From (16 a) assuming positive E_1^*,

$$- E_1^* \, S_{12}^{*(+)} \leq \nu_{12}^* \leq -E_1^* \, S_{12}^{*(-)} \quad . \tag{186}$$

If the left and right sides are positive, which is mostly the case, then because of (185)

$$- E_1^{*(-)} \, S_{12}^{*(+)} \leq \nu_{12}^* \leq -E_2^{*(+)} \, S_{12}^{*(-)} \quad . \tag{187}$$

Thus, (187) gives bounds on a Poisson's ratio.

Note that in a few cases, negative values of Poisson's ratio were obtained by numerical calculation (see Tables V to VII).

It is of utmost importance to compare the numerical values obtained in this report to experimental results. Unfortunately, consistent experimental results do not seem to be available in the literature.

A check with some preliminary experimental results (privately communicated) was given in Progress Report No. 8 under this contract. However, as pointed out there, the experimental results are inconsistent.

Table I

LOWER BOUNDS OF EFFECTIVE MODULI, C^*_{ij} (10^6 psi)
BORON/EPOXY, $G_t = G_t^{(-)}$, $\phi = 30°$

VF	C^*_{11}	C^*_{22}	C^*_{33}	C^*_{12}	C^*_{13}	C^*_{23}
0.1	4.06398	0.92263	1.27007	0.47054	1.51607	0.47850
0.2	7.33979	1.07215	1.76499	0.51625	2.60366	0.52991
0.3	10.63586	1.26332	2.29864	0.57258	3.69748	0.58937
0.4	13.96205	1.51632	2.88984	0.64528	4.79919	0.66262
0.5	17.33581	1.86696	3.57193	0.74467	5.91383	0.76029
0.6	20.79132	2.38527	4.41018	0.89091	7.05048	0.90310
0.7	24.40579	3.22934	5.55201	1.12925	8.22932	1.13755
0.8	28.39842	4.84713	7.41578	1.58744	9.50610	1.59548
0.9	33.73455	9.20441	11.84231	2.82342	11.11122	2.86432
1.0	63.88867	63.88846	63.88848	15.97217	15.97221	15.97224

TABLE II

UPPER BOUNDS OF EFFECTIVE MODULI, C^*_{ij} (10^6psi)

BORON/EPOXY, $G_t = G_t^{(+)}$, $\phi = 30°$

VF	C^*_{11}	C^*_{22}	C^*_{33}	C^*_{12}	C^*_{13}	C^*_{23}
0.1	4.06422	0.92646	1.27223	0.47150	1.51679	0.48137
0.2	7.34084	1.08908	1.77451	0.52049	2.60717	0.54260
0.3	10.63848	1.30515	2.32217	0.58304	3.70532	0.62074
0.4	13.96716	1.59803	2.93580	0.66571	4.81451	0.72391
0.5	17.34466	2.00846	3.65153	0.78005	5.94036	0.86641
0.6	20.80573	2.61589	4.53990	0.94856	7.09372	1.07606
0.7	24.42891	3.59916	5.76003	1.22170	8.29866	1.41491
0.8	28.43675	5.46029	7.76067	1.74073	9.[illegible]07	2.05535
0.9	33.80588	10.34575	12.48432	3.10875	11.32522	3.72024
1.0	63.88867	63.88840	63.88845	15.97215	15.97220	15.97219

TABLE III

EFFECTIVE MODULI, C^*_{ij} (10^6psi)

BORON/EPOXY, $G_t = G_t^{(1)}$, $\phi = 30°$

VF	C^*_{11}	C^*_{22}	C^*_{33}	C^*_{12}	C^*_{13}	C^*_{23}
0.1	4.06406	0.92390	1.27079	0.47086	1.51631	0.47946
0.2	7.34014	107780	1.76816	0.51767	2.60505	0.53414
0.3	10.63673	1.27726	2.30649	0.57607	3.70009	0.59983
0.4	13.96376	1.54356	2.90516	0.65209	4.80430	0.68305
0.5	17.33876	1.91413	3.59846	0.75646	5.92268	0.79566
0.6	20.79613	2.46214	4.45342	0.91012	7.06489	0.96076
0.7	24.41350	3.35262	5.62135	1.16006	8.25243	1.23000
0.8	28.41120	5.05152	7.53074	1.63854	9.54443	1.74877
0.9	33.75833	9,58485	12.05632	2.91853	11.18255	3.14957
1.0	63.88867	63.88844	63.88847	15.97216	15.97220	15.97222

TABLE IV

EFFECTIVE MODULI, C^*_{ij} (10^6psi)

BORON/EPOXY, $G_t = G_t^{(2)}$, $\phi = 30°$

VF	C^*_{11}	C^*_{22}	C^*_{33}	C^*_{12}	C^*_{13}	C^*_{23}
0.1	4.06414	0.92518	1.27151	0.47118	1.51655	0.48042
0.2	7.34049	1.08344	1.77134	0.51908	2.60611	0.53837
0.3	10.63760	1.29120	2.31433	0.57956	3.70271	0.61028
0.4	13.96546	1.57079	2.92048	0.65890	4.80941	0.70348
0.5	17.34171	1.96129	3.62499	0.76826	5.93152	0.83104
0.6	20.80093	2.53901	4.49666	0.92934	7.07930	1.01841
0.7	24.42120	3.47589	5.69069	1.19088	8.27554	1.32245
0.8	28.42397	5.25590	7.64571	1.68964	9.58275	1.90206
0.9	33.78210	9.96530	12.27032	3.01364	11.25389	3.43491
1.0	63.88867	63.88842	63.88846	15.97216	15.97220	15.97221

TABLE V

BOUNDS OF EFFECTIVE COMPLIANCE, $S^*_{ij} \times 10^6$

BORON/EPOXY, $G_t = G_t^{(+)}$, $\phi = 30°$

VF	S^*_{11}	S^*_{22}	S^*_{33}	S^*_{12}	S^*_{13}	S^*_{23}
0.1	0.44607	1.35195	1.66902	0.06138	-0.55506	-0.58472
0.2	0.29177	1.10945	1.37568	0.08746	-0.45542	-0.46774
0.3	0.21832	0.90572	1.11780	0.07808	-0.36923	-0.36669
0.4	0.17006	0.72741	0.89274	0.06247	-0.29430	-0.28181
0.5	0.13403	0.57101	0169685	0.04679	-0.22915	-0.21161
0.6	0.10532	0.43373	0.52601	0103269	-0.17233	-0.153890
0.7	0.081516	0.31269	0.37617	0.020477	-0.122472	-0.106310
0.8	0.061062	0.205066	0.243670	0.010028	-0.078356	-0.066742
0.9	0.042509	0.108294	0.125361	0.001225	-0.038927	-0.033382
1.0	0.017391	0.017391	0.017391	-0.003478	-0.003478	-0.003478

TABLE VI

BOUNDS OF EFFECTIVE COMPLIANCE, $S^*_{ij} \times 10^6$

BORON/EPOXY, $G_t = G_t^{(-)}$, $\phi = 30°$

VF	S^*_{11}	S^*_{22}	S^*_{33}	S^*_{12}	S^*_{13}	S^*_{23}
0.1	0.446308	1.355273	1.670097	0.060507	-0.555551	-0.582827
0.2	0.292335	1.119950	1.379173	0.085023	-0.456826	-0.461687
0.3	0.219223	0.924096	1.124055	0.074002	-0.371605	-0.355973
0.4	0.171229	0.751987	0.901319	0.057120	-0.297459	-0.267286
0.5	0.135325	0.598766	0.706788	0.40800	-0.232733	-0.194998
0.6	0.106599	0.461206	0.536071	0.026785	-0.175903	-0.137265
0.7	0.082613	0.336601	0.385070	0.015353	-0.125597	-0.091723
0.8	0.061857	0.222670	0.250267	0.006287	-0.080646	-0.055966
0.9	0.042905	0.117497	0.128776	-0.000686	-0.40091	-0.027775
1.0	0.017391	0.017391	0.017391	-0.003478	-0.003478	-0.003478

TABLE VII

EFFECTIVE COMPLIANCE, $S^*_{ij} \times 10^6$

BORON/EPOXY, $G_t = G_t^{(1)}$, $\phi = 30°$

VF	S^*_{11}	S^*_{22}	S^*_{33}	S^*_{12}	S^*_{13}	S^*_{23}
0.1	0.446231	1.354162	1.669738	0.060798	-0.555385	-0.583459
0.2	0.292144	1.116406	1.377995	0.085845	-0.456352	-0.463731
0.3	0.218914	0.917813	1.121917	0.075396	-0.370792	-0.359639
0.4	0.170824	0.743450	0.898339	0.058978	-0.296360	-0.272331
0.5	0.134868	0.588966	0.703278	0.042916	-0.231467	-0.200863
0.6	0.106144	0.451354	0.532464	0.028904	-0.174621	-0.143226
0.7	0.082214	0.327913	0.381838	0.017215	-0.124462	-0.097022
0.8	0.061566	0.216218	0.247849	0.007658	-0.079807	-0.059916
0.9	0.042760	0.114128	0.127526	0.000014	-0.039665	-0.029827
1.0	0.017391	0.017391	0.017391	-0.003478	-0.003478	-0.003478

TABLE VIII

EFFECTIVE COMPLIANCE, $S^*_{ij} \times 10^6$

BORON/EPOXY, $G_t = G_t^{(2)}$, $\phi = 30°$

VF	S^*_{11}	S^*_{22}	S^*_{33}	S^*_{12}	S^*_{13}	S^*_{23}
0.1	0.446155	1.353055	1.669379	0.061088	-0.555220	-0.584089
0.2	0.291956	1.112906	1.376830	0.086658	-0.455883	-0.465749
0.3	0.218613	0.911690	1.119833	0.076755	-0.369999	-0.363211
0.4	0.170436	0.735264	0.895480	0.060760	-0.295307	-0.277168
0.5	0.134437	0.579728	0.699970	0.04491	-0.230273	-0.206391
0.6	0.105721	0.442220	0.529121	0.030868	-0.173433	-0.148752
0.7	0.081850	0.319972	0.378884	0.018916	-0.123424	-0.101865
0.8	0.061302	0.210378	0.245661	0.008899	-0.79047	-0.063491
0.9	0.042629	0.111075	0.126393	0.000647	-0.039279	-0.031688
1.0	0.017391	0.017391	0.017391	-0.003478	-0.003478	-0.003478

TABLE IX

YOUNG'S MODULUS, $E_3 (10^6$ psi)

BORON/EPOXY, ϕ = 30°

VF	Upper Bound $G_t = G_t^{(+)}$	$G_t = G_t^{(2)}$	$G_t = G_t^{(1)}$	Lower Round $G_t = G_t^{(-)}$
0.1	0.5990	0.5992	0.5988	0.5989
0.2	0.7263	0.7269	0.7251	0.7257
0.3	0.8930	0.8946	0.8896	0.8913
0.4	1.1167	1.1202	1.1095	1.1132
0.5	1.4286	1.4350	1.4149	1.4219
0.6	1.8899	1.9011	1.8654	1.8781
0.7	2.6393	2.6583	2.5969	2.6189
0.8	4.0707	4.1039	3.9957	4.0347
0.9	7.9118	7.9770	7.7654	7.8415
1.0	57.5000	57.5000	57.5000	57.5000

TABLE X

POISSON'S RATIO, ν_{31},

BORON/EPOXY, $\phi = 30°$

VF	Upper Bound $G_t = G_t^{(+)}$	$G_t = G_t^{(2)}$	$G_t = G_t^{(1)}$	Lower Bound $G_t = G_t^{(-)}$
0.1	0.3326	0.3326	0.3326	0.3326
0.2	0.3312	0.3312	0.3311	0.3311
0.3	0.3305	0.3306	0.3303	0.3304
0.4	0.3299	0.3300	0.3297	0.3298
0.5	0.3291	0.3293	0.3288	0.3290
0.6	0.3279	0.3281	0.3276	0.3278
0.7	0.3260	0.3262	0.3256	0.3258
0.8	0.3220	0.3222	0.3216	0.3218
0.9	0.3110	0.3113	0.3105	0.3108
1.0	0.2000	0.2000	0.2000	0.2000

13. DETERMINATION OF BOUNDS ON THE SHELL STIFFNESS TENSOR FOR BIAXIALLY REINFORCED MATERIALS

GENERAL THEORY

We now derive expressions for the components of the shell stiffness tensor in terms of the moduli for biaxially reinforced material. It will be shown that bounds on the moduli correspond to bounds on the various components of the shell stiffness tensor. The initial discussion will be presented in tensor form, which is particularly convenient, and the final results for orthotropic shells of revolution will be presented in terms of appropriate physical components.

For shell theory, we use surface tensors, and components are denoted by Greek indices which take the values of 1, 2.

The constitutive relations for shells based on the Kirchhoff hypothesis take the form:

$$\tilde{N}^{\alpha\beta} = D_o^{\alpha\beta\nu\lambda}\, p_{\nu\lambda} + D_1^{\alpha\beta\nu\lambda}\, q_{\nu\lambda}, \tag{188}$$

$$M^{\alpha\beta} = D_1^{\nu\lambda\alpha\beta}\, p_{\nu\lambda} + D_2^{\nu\lambda\alpha\beta}\, q_{\nu\lambda}, \tag{189}$$

where $\tilde{N}^{\alpha\beta}$ and $M^{\alpha\beta}$ are suitably defined symmetric stress-resultants and stress couples, respectively; $p_{\nu\lambda}$ and $q_{\nu\lambda}$ are the strain measures of shell theory defined below; and the quantities $D_n^{\alpha\beta\nu\lambda}$ (n = 0,1,2) are the components of the shell stiffness tensor. The shell strain measures are given in terms of the components of displacement by the relations

$$p_{\alpha\beta} = \frac{1}{2}\left[V_{\alpha\|\beta} + V_{\beta\|\alpha} - \left(b_{\alpha\beta} + b_{\beta\alpha}\right)w\right], \tag{190}$$

$$q_{\alpha\beta} = \frac{1}{2}\left[\beta_{\alpha\|\beta} + \beta_{\beta\|\alpha} - b_\alpha^\nu V_{\alpha\|\beta} - b_\beta^\nu V_{\nu\|\alpha} + \left(b_\alpha^\nu b_{\nu\beta} + b_\beta^\nu b_{\nu\alpha}\right)w\right]. \tag{191}$$

In equations (190) and (191), V_α are the membrane displacements, while w is the normal displacement; β_α represents the components of the rotation of the normal to the middle surface, and the notation $()_{\|\alpha}$ indicates covariant differentiation with respect to the surface metric. The quantities $b_{\alpha\beta}$ and b_β^α are coefficients of the second fundamental form of the middle surface and are related to the principal radii of curvature of the surface.

By virtue of the neglect of transverse shear deformation, the rotation components β_α are related to the displacements by the relations

$$\beta_\alpha = -\frac{\partial w}{\partial x_\alpha} - b_\alpha^\nu v_\nu . \tag{192}$$

The shell stiffness tensors $D_n^{\alpha\beta\nu\lambda}$ appearing in Equations (188) and (189) are defined by considering the strain energy function W for the shell and requiring that the following relation be satisfied:

$$\tilde{N}^{\alpha\beta} = \frac{\partial W}{\partial p_{\alpha\beta}} , \quad M^{\alpha\beta} = \frac{\partial W}{\partial q_{\alpha\beta}} . \tag{193}$$

This insures that the constitutive relations of shell theory are consistent with the laws of thermodynamics. The shell strain energy density function is then written in the form

$$2W = D_o^{\nu\lambda\epsilon\kappa} p_{\nu\lambda} p_{\epsilon\kappa} + 2D_1^{\epsilon\kappa\nu\lambda} q_{\nu\lambda} p_{\epsilon\kappa} + D_2^{\nu\lambda\epsilon\kappa} q_{\nu\lambda} q_{\epsilon\kappa} . \tag{194}$$

By virtue of the symmetry of the stress resultants and couples and of the strain measures, as well as the positive definite character of W, it can be shown that the various stiffness tensors satisfy the following symmetries:

$$\begin{aligned} D_o^{\alpha\beta\gamma\delta} &= D_o^{\beta\alpha\gamma\delta} = D_o^{\alpha\beta\delta\gamma} = D_o^{\gamma\delta\alpha\beta} , \\ D_2^{\alpha\beta\gamma\delta} &= D_2^{\beta\alpha\gamma\delta} = D_2^{\alpha\beta\delta\gamma} = D_2^{\gamma\delta\alpha\beta} , \\ D_1^{\alpha\beta\gamma\delta} &= D_1^{\beta\alpha\gamma\delta} = D_1^{\alpha\beta\delta\gamma} . \end{aligned} \tag{195}$$

It should be noted that the tensor $D_1^{\alpha\beta\gamma\delta}$ does not exhibit symmetry with respect to the pairs of indices $(\alpha\beta)$ and $(\gamma\delta)$.

The derivation leading to the definition of the stiffness tensor can be found in [9] and will not be repeated here. The components of $D_n^{\nu\lambda\epsilon\kappa}$ are given in the equation:

$$\begin{aligned} D_o^{\nu\lambda\epsilon\kappa} = E_o^{\nu\lambda\epsilon\kappa} + \frac{1}{2}\Big[& b_\alpha^\kappa \left(B_1^{\nu\lambda\epsilon\alpha} + B_1^{\lambda\nu\epsilon\alpha}\right) + b_\alpha^\epsilon \left(B_1^{\nu\lambda\kappa\alpha} + B_1^{\lambda\nu\kappa\alpha}\right) \\ & + b_\alpha^\lambda \left(B_1^{\nu\alpha\epsilon\kappa} + B_1^{\nu\alpha\kappa\epsilon}\right) + b_\alpha^\nu \left(B_1^{\lambda\alpha\epsilon\kappa} + B_1^{\lambda\alpha\kappa\epsilon}\right)\Big] , \end{aligned} \tag{196}$$

$$D_1^{\epsilon\kappa\nu\lambda} = \frac{1}{2}\left[B_1^{\epsilon\kappa\nu\lambda} + B_1^{\epsilon\kappa\lambda\nu} + b_\gamma^\epsilon \left(B_2^{\kappa\gamma\nu\lambda} + B_2^{\kappa\gamma\lambda\nu}\right)\right] , \tag{197}$$

$$D_2^{\nu\lambda\epsilon\kappa} = \frac{1}{4}\left[B_2^{\nu\lambda\epsilon\kappa} + B_2^{\nu\lambda\kappa\epsilon} + B_2^{\lambda\nu\epsilon\kappa} + B_2^{\lambda\nu\kappa\epsilon}\right] . \tag{198}$$

In Equations (196) through (198), the various quantities are defined as

$$E_o^{\nu\lambda\epsilon\kappa} = \int_{-h/2}^{h/2} \mu \bar{A}^{\nu\lambda\epsilon\kappa} \, dx^3, \tag{199}$$

$$B_n^{\nu\lambda\epsilon\kappa} = \int_{-h/2}^{h/2} \bar{A}^{\nu\pi\epsilon\rho} \, \mu \bar{\mu}_\pi^{\lambda} \bar{\mu}_\rho^{\kappa} \, (x^3)^n \, dx^3 \quad (n = 0,1,2). \tag{200}$$

In these equations, we have introduced the shifters μ_β^α defined as

$$\mu_\beta^\alpha = \delta_\beta^\alpha - x^3 b_\mu^\alpha , \tag{201}$$

while the "inverse shifters" are defined by the equation

$$\mu_\beta^\alpha \bar{\mu}_\gamma^\beta = \delta_\gamma^\alpha , \tag{202}$$

and we have used the notation

$$\mu = |\mu_\beta^\alpha| . \tag{203}$$

In Equations (199) and (200), the quantities $\bar{A}^{\nu\lambda\epsilon\kappa}$ are the shifted components of the elasticity tensor for the material. The term "shifted" indicates that these components are associated with a system of base vectors consisting of the middle surface base vectors and a unit vector normal to the middle surface.

It should be noted that the integrations indicated in Equations (199) and (200) can be carried out exactly, even though the shifters $\bar{\mu}_\beta^\alpha$ are functions of the thickness coordinate x^3. However, for most cases of practical interest, it is permissable to neglect terms of order $\frac{h}{R}$ in comparison to unity, where h denotes the shell thickness and R the least radius of curvature. The result of such a procedure is a first-order shell theory which is sufficiently accurate for most cases. In the remainder of the present work, we will restrict our consideration to this first-order theory. Furthermore, to simplify the results we assume that the layers are located symmetrically with respect to the shell reference surface. Introducing the approximation $\frac{h}{R} << 1$ in Equations (196) through (200) leads to the following results for a single-layer shell:

$$D_o^{\alpha\beta\gamma\delta} = \bar{A}^{\alpha\beta\gamma\delta}\, h,$$

$$D_1^{\alpha\beta\gamma\delta} = 0\ ,$$

$$D_2^{\alpha\beta\gamma\delta} = \bar{A}^{\alpha\beta\gamma\delta}\, \frac{h^3}{12}\ . \tag{204}$$

Thus, the constitutive relations are now reduced to the form

$$\tilde{N}^{\alpha\beta} = D_o^{\alpha\beta\gamma\delta}\, p_{\gamma\delta} = \bar{A}^{\alpha\beta\gamma\delta}\, hp_{\gamma\delta}\ ,$$

$$M^{\alpha\beta} = D_o^{\alpha\beta\gamma\delta}\, p_{\gamma\delta} = \bar{A}^{\alpha\beta\gamma\delta}\, \frac{h^3}{12}\, q_{\gamma\delta}\ . \tag{205}$$

It is clear from Equations (204) that, if we have bounds on the components of the elasticity tensor $\bar{A}^{\alpha\beta\gamma\delta}$, we have corresponding bounds on the shell stiffness tensor. The fact that we are using the shifted components of the elasticity tensor does not alter this property, since the "shifting" operation depends only on the geometry of the coordinate system chosen.

CONSTITUITIVE RELATIONS FOR ORTHOTROPIC SHELLS

For an orthotropic material with planes of elastic symmetry coinciding with the coordinate planes, the stress-strain relations may be written in the form

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{23} \\ \sigma^{31} \end{bmatrix} = \begin{bmatrix} C_{11}^{11} & C_{22}^{11} & C_{33}^{11} & 0 & 0 & 0 \\ C_{11}^{22} & C_{22}^{22} & C_{33}^{22} & 0 & 0 & 0 \\ C_{11}^{33} & C_{22}^{33} & C_{33}^{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{12}^{12} & 0 & 0 \\ 0 & 0 & 0 & & C_{23}^{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{31}^{31} \end{bmatrix} \begin{bmatrix} e^{11} \\ e^{22} \\ e^{33} \\ e^{12} \\ e^{23} \\ e^{31} \end{bmatrix}, \tag{206}$$

where the physical components of the stress, strain, and elasticity tensors have been used. Thus, the quantities $C_{k\ell}^{ij}$ are precisely the moduli previously bounded for the biaxially reinforced material. Since the classical theory of shells neglects the effect of transverse normal stress and tranverse shear deformation, we are only interested in those parts of Equation (206)which refer to the 1 and 2 directions. Therefore, for our

purposes, we may write a restricted form of Equation (206) as follows:

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{12} \end{bmatrix} = \begin{bmatrix} C^{11}_{11} & C^{11}_{22} & 0 \\ C^{22}_{11} & C^{22}_{22} & 0 \\ 0 & 0 & C^{12}_{12} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \end{bmatrix} . \tag{207}$$

We note that the moduli in Equations (206) and (207) satisfy the relations $C^{ij}_{k\ell} = C^{k\ell}_{ij}$ so that the matrices are symmetric.

Now, the shifted components of the elasticity tensor in orthogonal surface coordinates are given by

$$\bar{A}^{\alpha\alpha\beta\beta} = C^{\alpha\alpha}_{\beta\beta}\, a^{\beta\beta}\, a^{\beta\beta} \qquad \text{(no sum on } \alpha,\beta) ,$$

$$\bar{A}^{\alpha\beta\alpha\beta} = C^{\alpha\beta}_{\alpha\beta}\, a^{\alpha\alpha}\, a^{\beta\beta} \qquad \text{(no sum on } \alpha,\beta) , \tag{208}$$

where $a_{\alpha\beta}$ are the components of the surface metric and the $a^{\alpha\beta}$ are the corresponding components of the conjugate tensor, defined such that

$$a_{\alpha\beta}\, a^{\beta\gamma} = \delta^{\gamma}_{\alpha} . \tag{209}$$

We note that for orthogonal surface coordinates, we have $a_{\alpha\beta} = 0$ $(\alpha \neq \beta)$, and it follows that

$$a^{\alpha\alpha} = \frac{1}{a_{\alpha\alpha}} \qquad \text{(no sum)}. \tag{210}$$

In order to identify the various components of the shell stiffness tensor in terms of the previously bounded moduli for biaxially reinforced material, it is convenient to rewrite Equation (205) in matrix form in terms of the physical components of the various tensors. When this is done, we note that for an orthotropic material where planes of symmetry coincide with the lines of curvature of the middle surface, the stress-strain relations may be separated into two groups involving normal and shear components, respectively. The resulting equation may then be written in the form

$$
\begin{bmatrix} \tilde{N}^{<11>} \\ \tilde{N}^{<22>} \\ M^{<11>} \\ M^{<22>} \end{bmatrix} = \begin{bmatrix} D_o^{1111} a_{11} a_{11} & D_o^{2211} a_{11} a_{22} & 0 & 0 \\ D_o^{2211} a_{11} a_{22} & D_o^{2222} a_{22} a_{22} & 0 & 0 \\ 0 & 0 & D_2^{1111} a_{11} a_{11} & D^{1122} a_{11} a_{22} \\ 0 & 0 & D_2^{2211} a_{11} a_{22} & D_2^{2222} a_{22} a_{22} \end{bmatrix} \begin{bmatrix} p_{<11>} \\ p_{<22>} \\ q_{<11>} \\ q_{<22>} \end{bmatrix} , \tag{211}
$$

$$
\begin{bmatrix} \tilde{N}^{<12>} \\ \tilde{M}^{<12>} \end{bmatrix} = \begin{bmatrix} 2D_o^{1212} a_{11} a_{22} & 0 \\ 0 & 2D_2^{1212} a_{11} a_{22} \end{bmatrix} \begin{bmatrix} p_{<12>} \\ q_{<12>} \end{bmatrix} , \tag{212}
$$

where the symbol <> denotes physical components.

Substituting Equation (204) into Equations (211) and (212) and making use of Equations (208) through (210) yields the required stress-strain relations in terms of the previously bounded meduli; the result is

$$
\begin{bmatrix} \tilde{N}^{<11>} \\ \tilde{N}^{<22>} \\ M^{<11>} \\ M^{<22>} \end{bmatrix} = \begin{bmatrix} C_{11}^{11} h & C_{11}^{22} h & 0 & 0 \\ C_{22}^{11} h & C_{22}^{22} h & 0 & 0 \\ 0 & 0 & C_{11}^{11} \frac{h^3}{12} & C_{11}^{22} \frac{h^3}{12} \\ 0 & 0 & C_{22}^{11} \frac{h^3}{12} & C_{22}^{22} \frac{h^3}{12} \end{bmatrix} \begin{bmatrix} p_{<11>} \\ p_{<22>} \\ q_{<11>} \\ q_{<22>} \end{bmatrix} , \tag{213}
$$

$$
\begin{bmatrix} \tilde{N}^{<12>} \\ M^{<12>} \end{bmatrix} = \begin{bmatrix} 2C_{12}^{12} h & 0 \\ 0 & 2C_{12}^{12} \frac{h^3}{12} \end{bmatrix} \begin{bmatrix} p_{<12>} \\ q_{<12>} \end{bmatrix} . \tag{214}
$$

We see from Equations (213) and (214) that we need to evaluate the following stiffnesses:

$$C^{11}_{11}h,\ C^{11}_{11}\frac{h^3}{12},\ C^{22}_{11}h = C^{11}_{22}h,$$

$$C^{22}_{11}\frac{h^3}{12} = C^{11}_{22}\frac{h^3}{12},\ C^{22}_{22}h,\ C^{22}_{22}\frac{h^3}{12},\ C^{12}_{12}h,\ C^{12}_{12}\frac{h^3}{12} \tag{215}$$

In terms of the six-by-six matrix notation employed in previous chapters, the moduli $C^{\alpha\beta}_{\gamma\delta}$ are identified from the equation

$$\begin{bmatrix} C^{11}_{11} & C^{22}_{11} & C^{33}_{11} & 0 & 0 & 0 \\ C^{11}_{22} & C^{22}_{22} & C^{33}_{22} & 0 & 0 & 0 \\ C^{11}_{33} & C^{22}_{33} & C^{33}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C^{12}_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & C^{23}_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & C^{31}_{31} \end{bmatrix} = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{13} & 0 & 0 & 0 \\ C^*_{21} & C^*_{22} & C^*_{23} & 0 & 0 & 0 \\ C^*_{31} & C^*_{32} & C^*_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C^*_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C^*_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C^*_{66} \end{bmatrix} . \tag{216}$$

It is now clear from Equation (216) that bounds on the shell stiffness tensor are obtained from bounds on the moduli C^*_{ij} ($i, j = 1...6$) as previously obtained in Section 11. We note from equations (213) through (216) that if bounds on the quantities C^*_{11}, C^*_{12}, C^*_{22}, C^*_{44} are known, corresponding bounds on all nonvanishing components of the shell stiffness tensor are immediately known.

14. ALTERNATE SOLUTION FOR BOUNDS ON COMPONENTS OF THE SHELL STIFFNESS TENSOR

In Section 13, bounds on the components of the shell stiffness tensor were obtained in terms of the bounds previously determined for the moduli of biaxially reinforced materials.

This result was obtained by considering the shell as homogeneous and constructed from a material having the appropriate moduli for a biaxially reinforced material. In the present section, we present an alternate approach based on the use of a layered shell model hereafter designated by LSM.

In the LSM analysis, the shell is considered as constructed from m layers of uniaxially reinforced material with fibers oriented at angles ϕ_i ($i = 1,2,---m$) with respect to the lines of curvature of the shell reference surface. We again consider a classical shell theory under the Kirchhoff hypothesis and limit the analysis to the first-order shell theory ($\frac{h}{R} << 1$). Under these conditions, the components of the shell stiffness tensors are given by

$$D_n^{\alpha\beta\gamma\delta} = \sum_{i=1}^{m} \bar{A}_i^{\alpha\beta\gamma\delta} \left[\frac{\xi^{n+1}}{n+1}\right]_{\xi_i}^{\xi_{i+1}} \quad . \quad (n = 0,2) , \tag{217}$$

where ξ is the distance of any point from the reference surface, ξ_i denotes the distance from the reference surface to the inner surface of the i^{th} layer, and $\bar{A}_i^{\alpha\beta\gamma\delta}$ are shifted components of the elasticity tensor for the i^{th} layer. When Equation (217) is transformed to physical components, we obtain the following results

$$D_o^{<\alpha\beta\gamma\delta>} = \sum_{i=1}^{m} C_i{}^{\alpha\beta}_{\gamma\delta} (\xi_{i+1} - \xi_i) \quad , \tag{218}$$

$$D_2^{<\alpha\beta\gamma\delta>} = \sum_{i=1}^{m} C_i{}^{\alpha\beta}_{\gamma\delta} \frac{1}{3} (\xi_{i+1}^3 - \xi_i^3) \quad . \tag{219}$$

In Equations (218) and (219), the quantities $C_i{}^{\alpha\beta}_{\gamma\delta}$ are the elastic moduli of the layer referenced to the x_1, x_2, x_3 coordinate system of Figure 1. These moduli are those of a uniaxially reinforced material after transformation to the proper coordinate system and are identical to the quantities $'C_{ijkl}$ or $''C_{ijkl}$ of Equations (75) and (76).

It is convenient to express Equations (218) and (219) in terms of the thickness h_i of the i^{th} layer and the coordinate ξ_i, which will represent the distance from the reference surface to the middle surface of the i^{th} layer. In terms of these quantities, (218) and (219) become

$$D_o^{<\alpha\beta\gamma\delta>} = \sum_{i=1}^{m} C_{i\,\gamma\delta}^{\alpha\beta} h_i \,, \tag{220}$$

$$D_2^{<\alpha\beta\gamma\delta>} = \sum_{i=1}^{m} C_{i\,\gamma\delta}^{\alpha\beta} \left[\frac{h_i^3}{12} + \xi_i^2 h_i\right] . \tag{221}$$

Equations (220) and (221) may be used to calculate the shell stiffness tensor for the LSM for any configuration of layers, provided that the transformation of coordinates for each layer is performed using the appropriate angle ϕ_i in the transformation equations of Section 6. We note that the shell will exhibit complete anisotropy in the general case.

We now consider the special case of a biaxially reinforced shell having an even number of layers. We assume that the layers are arranged on either side of the shell middle surface so that if the layer with middle surface at $+\xi_i$ has a fiber angle ϕ_i, the layer with middle surface at $-\xi_i$ has a fiber angle $(180-\phi_i)$. Under these conditions, the elastic moduli for the two layers become $'C_{i\,\gamma\delta}^{\alpha\beta}$ and $''C_{i\,\gamma\delta}^{\alpha\beta}$, which satisfy the symmetry conditions (78). Because of symmetry of Equations (220, 221) with respect to ξ_i, we can then express the shell stiffness tensors as

$$D_o^{<\alpha\beta\gamma\delta>} = \sum_{i=1}^{m} \tilde{C}_{i\,\gamma\delta}^{\alpha\beta} h_i \,, \tag{222}$$

$$D_2^{<\alpha\beta\gamma\delta>} = \sum_{i=1}^{m} \tilde{C}_{i\,\gamma\delta}^{\alpha\beta} \left[\frac{h_i^3}{12} + \xi_i^2 h_i\right] , \tag{223}$$

where

$$\tilde{C}_{i\,\gamma\delta}^{\alpha\beta} = \frac{1}{2}\left('C_{i\,\gamma\delta}^{\alpha\beta} + ''C_{i\,\gamma\delta}^{\alpha\beta}\right) . \tag{224}$$

Note that Equation (224) is identical to (79). We note that, because of the symmetries outlined in Section 6, the only nonvanishing components of $\tilde{C}_{i\ \gamma\delta}^{\ \alpha\beta}$ are

$$\tilde{C}_{i\ 11}^{\ 11},\ \tilde{C}_{i\ 22}^{\ 22},\ \tilde{C}_{i\ 22}^{\ 11} = \tilde{C}_{i\ 11}^{\ 22},\ \tilde{C}_{i\ 12}^{\ 12} \quad . \tag{225}$$

Recall that the Greek indices do not take on the value of 3. Therefore, this type of construction gives rise to an orthotropic shell, as should be expected from the results of Sections 11 and 13. The constitutive relations for shells based on the LSM are then written in a form analogous to (213) and (214):

$$\begin{bmatrix} \tilde{N}^{<11>} \\ \tilde{N}^{<22>} \\ M^{<11>} \\ M^{<22>} \end{bmatrix} = \begin{bmatrix} D_o^{<1111>} & D_o^{<1122>} & 0 & 0 \\ D_o^{<1122>} & D_o^{<1122>} & 0 & 0 \\ 0 & 0 & D_2^{<1111>} & D_2^{<1122>} \\ 0 & 0 & D_2^{<1122>} & D_2^{<2222>} \end{bmatrix} \begin{bmatrix} p_{<11>} \\ p_{<22>} \\ q_{<11>} \\ q_{<22>} \end{bmatrix}, \tag{226}$$

$$\begin{bmatrix} \tilde{N}^{<12>} \\ M^{<12>} \end{bmatrix} = \begin{bmatrix} 2D_o^{<1212>} & 0 \\ 0 & 2D_2^{<1212>} \end{bmatrix} \begin{bmatrix} p_{<12>} \\ q_{<12>} \end{bmatrix}, \tag{227}$$

where the stiffness components are given in Equations (222) through (224). In the notation of Section 6 (see (81)), the moduli appearing in (225) can be identified in the six-by-six matrix notation according to scheme

$$\tilde{C}_{\ 11}^{\ 11} = \tilde{C}_{\ 11},\ \tilde{C}_{\ 22}^{\ 22} = \tilde{C}_{\ 22},\ \tilde{C}_{\ 22}^{\ 11} = \tilde{C}_{\ 12},\ \tilde{C}_{\ 12}^{\ 12} = \tilde{C}_{44}\ , \tag{228}$$

and the quantities $\tilde{C}_{ij}$ are given by Equations (82) in terms of the single-layer moduli C'_{ij} of Equations (42) and (43). Since four of these moduli are known exactly and the fifth is bounded as a result of the work of Hashin and Rosen [1], it is clear that an argument similar to that of Section 11, leading to (175) can be applied to obtain bounds for the shell stiffness tensors as defined by the LSM.

Therefore, we have succeeded in obtaining two different sets of bounds for the shell stiffness tensors. It is likely that both of these sets of bounds are in less agreement because of the manner in which the moduli bounds for the biaxially reinforced material were developed.

We note that the results obtained in Section 13 and 14 proceed from the use of a first-order shell theory based on the Kirchhoff hypothesis. Therefore, the transverse shear deformation and transverse normal stress have been neglected, and the application should be limited to sufficiently thin shells. When applied to such shells, these results should yield acceptable results for deformations as well as overall stress resultants and stress couples. Certain quantities such as the interlayer shear stresses cannot be calculated directly from such a theory but can be determined approximately after the displacement field is known. Thus, the results presented in Sections 13 and 14 are subject to the approximations of a first-order classical shell theory and are consistent with these approximations.

15. CONCLUSION

Bounds for the orthotropic effective elastic moduli and compliances of biaxially fiber reinforced materials have been derived by variational procedures. It has been assumed that the single-layer uniaxially reinforced layers are described by the composite-cylinder assemblage model, [1].

Numerical evaluation of the bounds showed that they are extremely close together in the majority of cases. Thus, they should provide good estimates for elastic properties of biaxially reinforced materials. The results obtained for effective elastic properties have also been exploited to construct bounds for the membrane and bending stiffnesses of orthotropic biaxially reinforced shells.

It is of utmost importance to compare the present theoretical results with experimental findings. Unfortunately, consistent experimental results for elastic properties of biaxially reinforced materials do not seem to be available in the literature. There is accordingly great need for such an experimental program.

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13 ABSTRACT

Upper and lower bounds on the effective elastic properties of biaxially fiber reinforced material are derived. Two geometric models for the composite have been considered, and results are presented for several practical materials. In addition, bounds on the components of the shell stiffness tensor are obtained in terms of the bounds of the elastic moduli.

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